PHYS 141/241

Lecture 05: Gravitational Potential and N-Body Equations

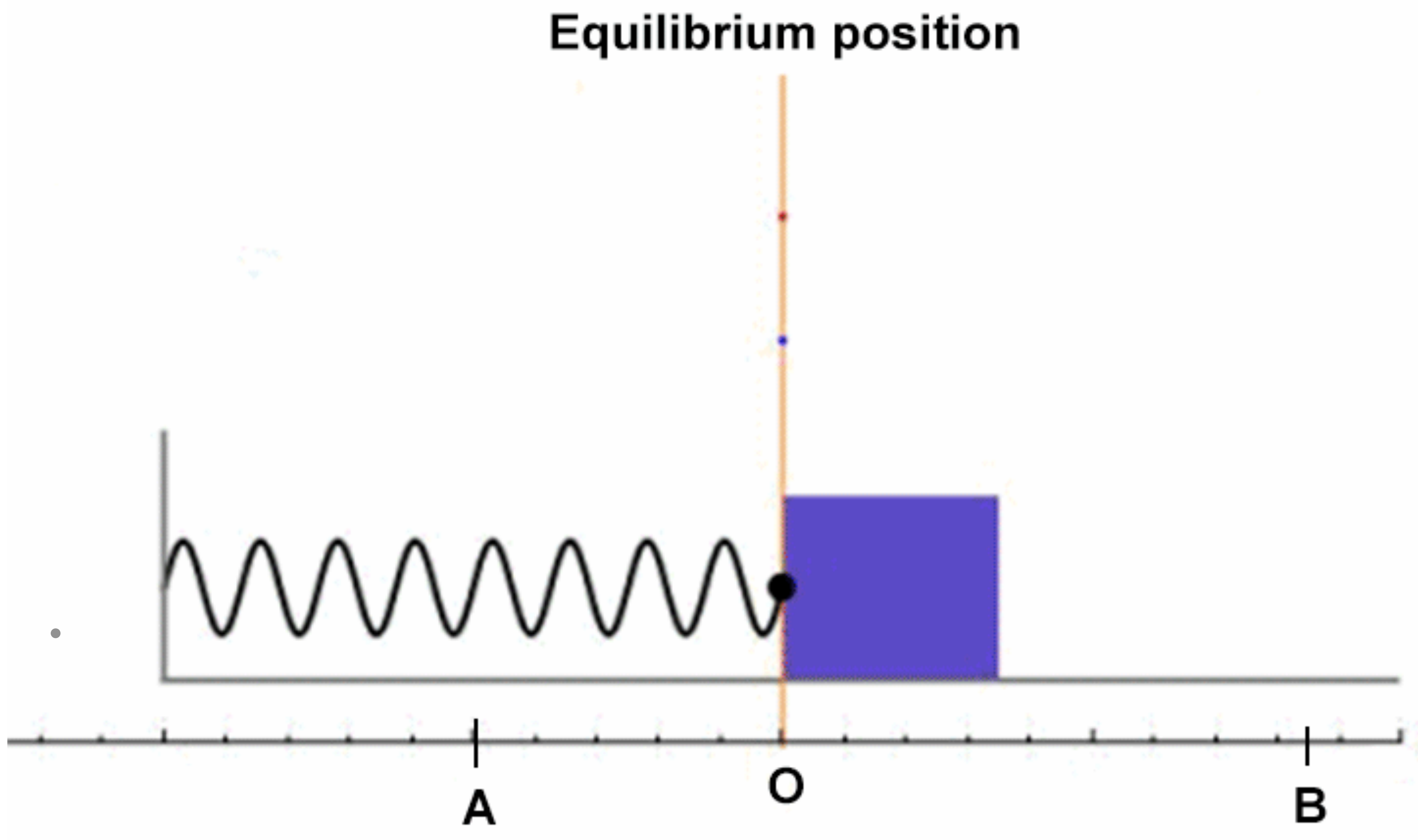
Javier Duarte — April 12, 2023

Conservative force fields (1D)

- In a 1D system, can define potential energy as

$$U(x) = -\int_{x_0}^{x} dx'\, f(x')$$

  - Note: $U(x_0) = 0$
  - Different choices of x_0 produce produce a different zero-point
- Example: simple harmonic oscillator like a spring



$$f(x) = -kx$$

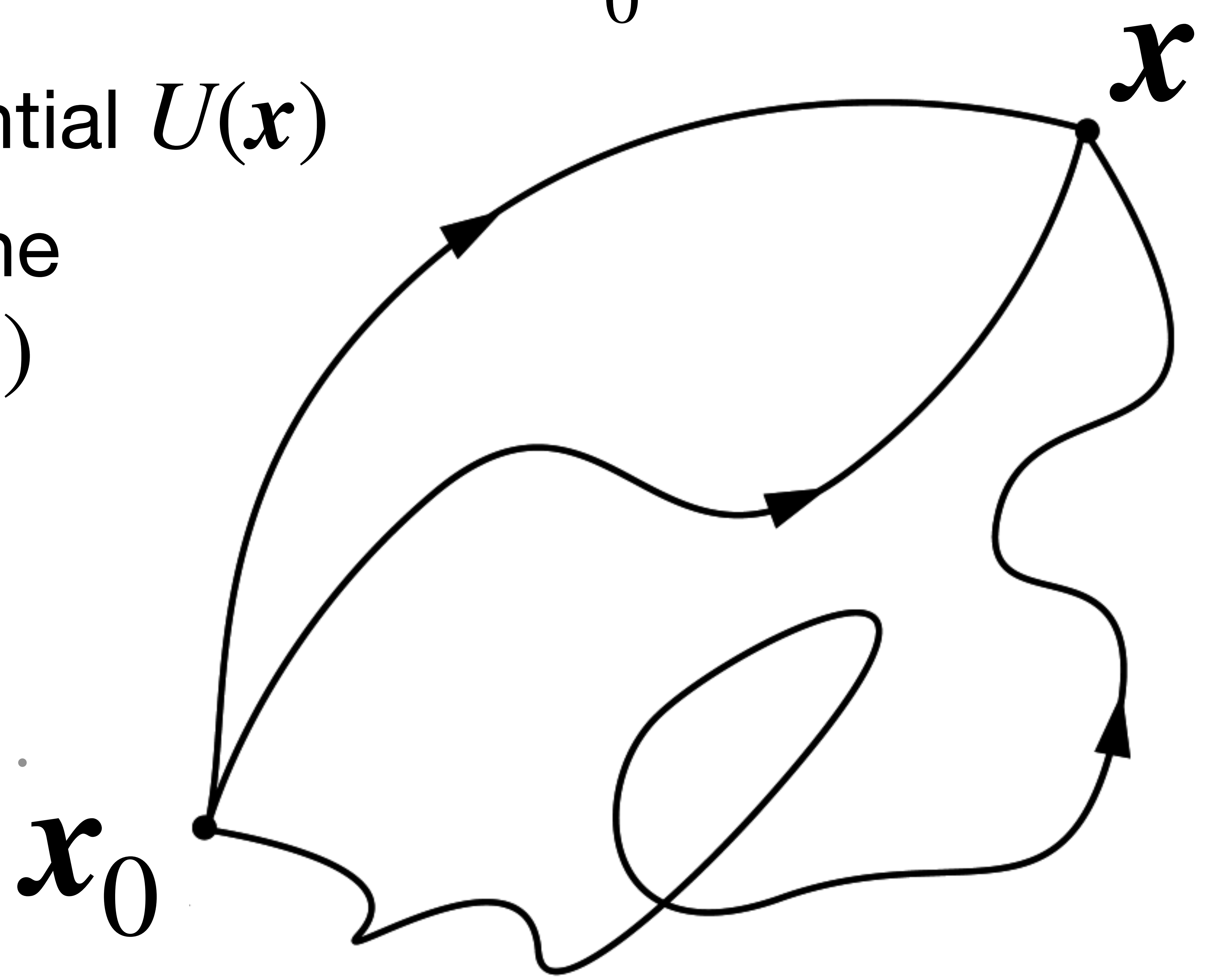
$$U(x) = -\int_{0}^{x} dx'\,(-kx') = \frac{1}{2}kx^2$$

Conservative force fields (nD)

- In $n > 1$ dimensions, we have the line integral

$$U(\boldsymbol{x}) = -\int_{\boldsymbol{x}_0}^{\boldsymbol{x}} d\boldsymbol{x}'\, f(\boldsymbol{x}')$$

- In general this may depend on the on the exact route taken from $\boldsymbol{x}_0$ to $\boldsymbol{x}$
  - If it does, then we cannot define a unique potential $U(\boldsymbol{x})$
  - One way to guarantee this is well defined is if the integral around any closed path of the force $f(\boldsymbol{x})$ vanishes
  - Equivalent condition: $f(\boldsymbol{x}) = -\nabla U(\boldsymbol{x})$
- Force fields sasyfing these conditions, e.g. gravitational field of a stationary point mass, are **conservative**

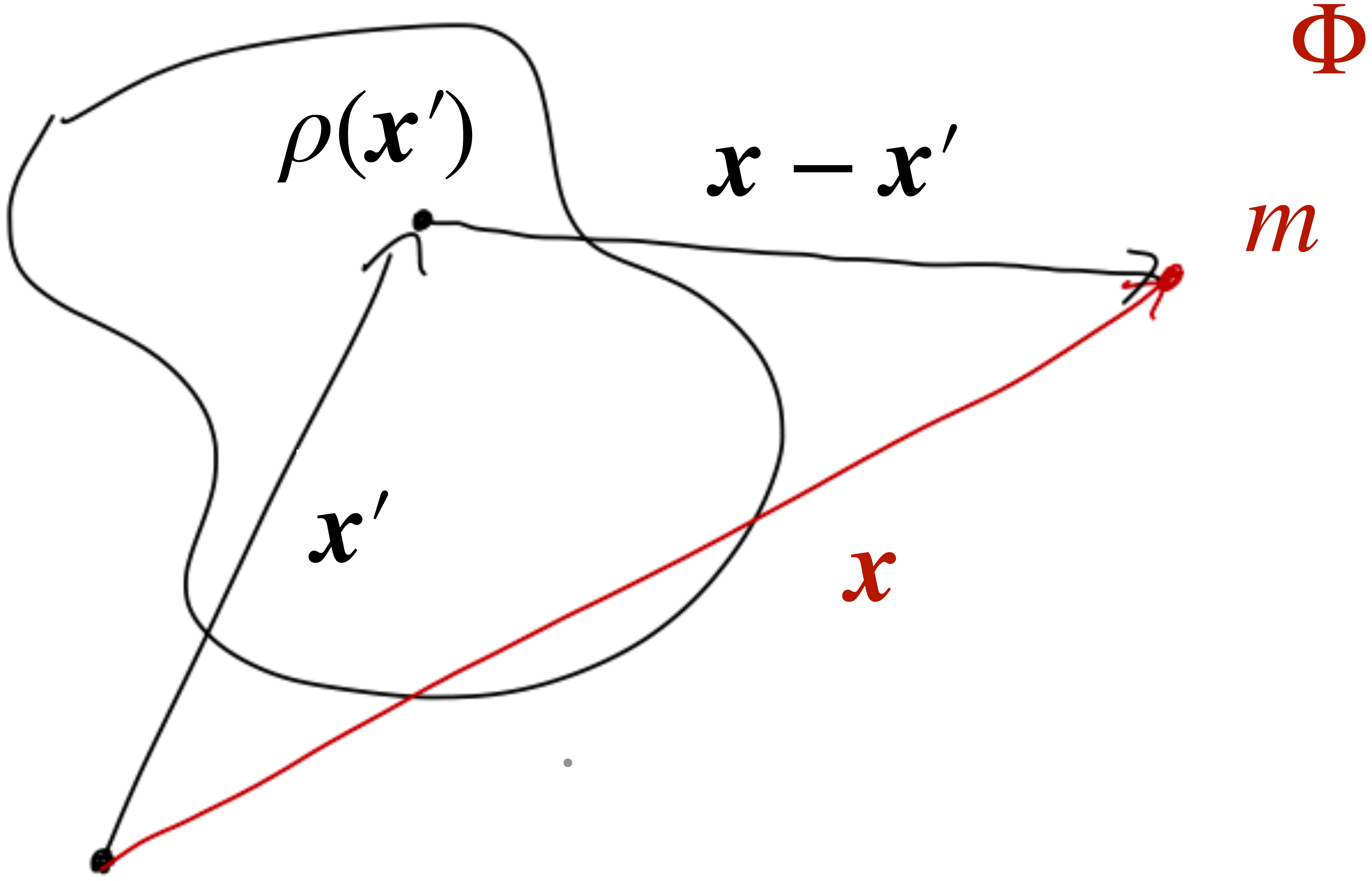


Gravitational potential

- For simulation, natural to work with the line integral of the acceleration rather than the force: *potential energy per unit mass* or *gravitational potential* $\Phi(\boldsymbol{x}')$
    - Given a point (or *test*) mass m, the potential energy is $U(\boldsymbol{x}') = m\Phi(\boldsymbol{x}')$
- For an arbitrary mass density $\rho(\boldsymbol{x}')$, the gravitational potential is

$$\Phi(\boldsymbol{x}) = -G\int d^3\boldsymbol{x}' \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x}' - \boldsymbol{x}|}$$

What is $\Phi(\boldsymbol{x})$ here?



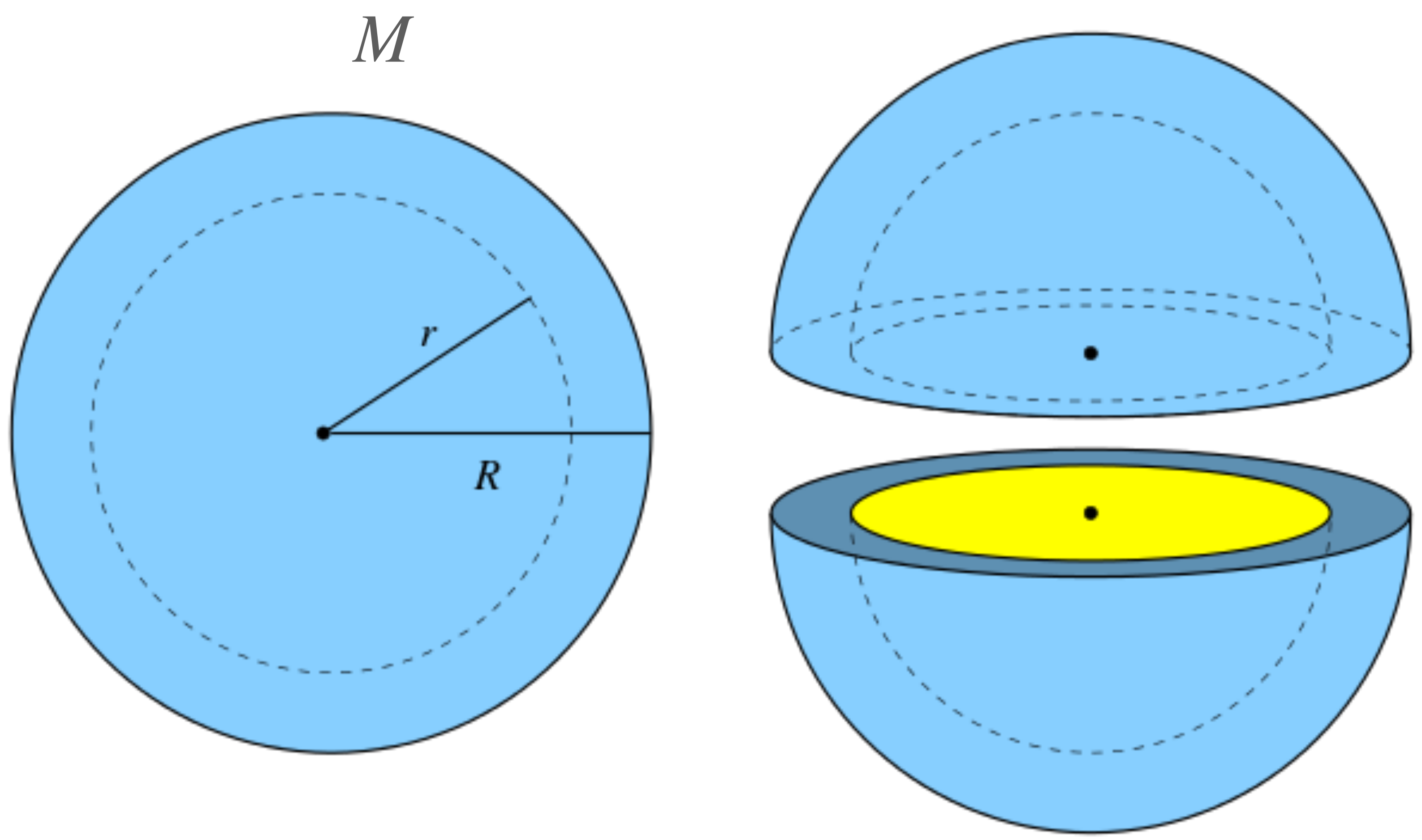
Gravitational potential (local)

- Equivalently, Poisson's equation relates the mass density and gravitational potential

$$\nabla^2\Phi(\boldsymbol{x}) = 4\pi G\rho(\boldsymbol{x}) \qquad (\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)$$

- This is linear! If $\rho_1(\boldsymbol{x})$ generates $\Phi_1(\boldsymbol{x})$ and $\rho_2(\boldsymbol{x})$ generates $\Phi_2(\boldsymbol{x})$, then $\rho_1(\boldsymbol{x}) + \rho_2(\boldsymbol{x})$ generates $\Phi_1(\boldsymbol{x}) + \Phi_2(\boldsymbol{x})$

Spherical potentials



- Consider an (infinitesimally thin) spherical shell of mass M radius R
  - The acceleration inside the shell vanishes
  - The acceleration outside the shell is $-GM/r^2$
- It follows that the gravitational potential of any spherical mass distribution is

$$\Phi(r) = \int_{r_0}^{r} dr' \ a(r') = G \int_{r_0}^{r} dr' \ \frac{M(r')}{r'^2}$$

- Where the enclosed mass is:

$$M(r) = 4\pi \int_{r_0}^{r} dr' \ r'^2 \rho(r')$$

Spherical potentials: examples

- Point of mass M:

  $$\Phi(r) = -G\frac{M}{r} \quad \text{(Keplerian potential)}$$

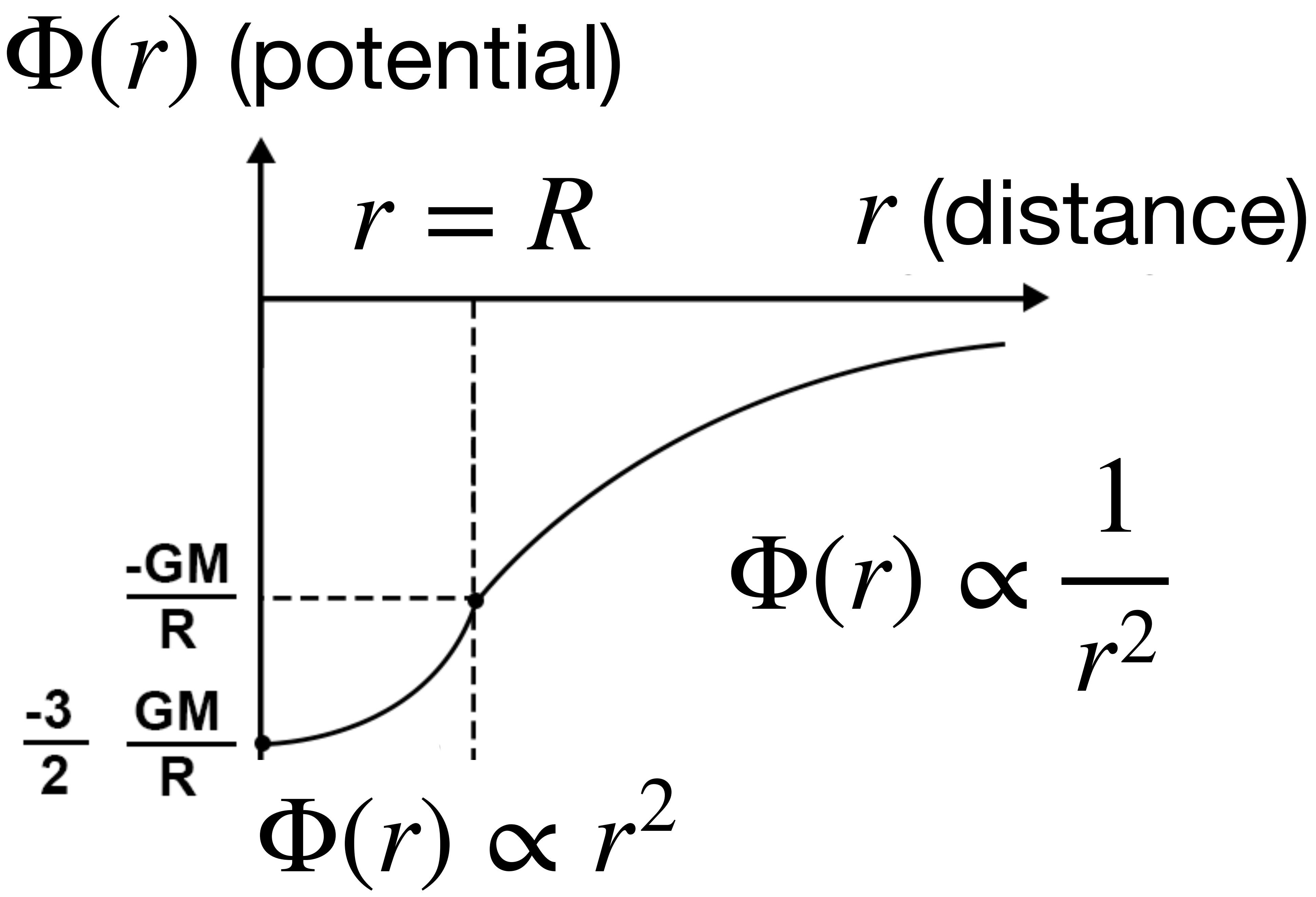
  - Orbits in this potential obey Kepler's 3 laws
  - If $G = 1$, the velocity of a circular orbit at radius r is: $v_c(r) = \sqrt{\frac{M}{r}}$

- Uniform sphere of mass M and radius R:

  $$\Phi(r) = -2\pi G\rho\left(R^2 - \frac{r^2}{3}\right) \qquad r \leq R,$$

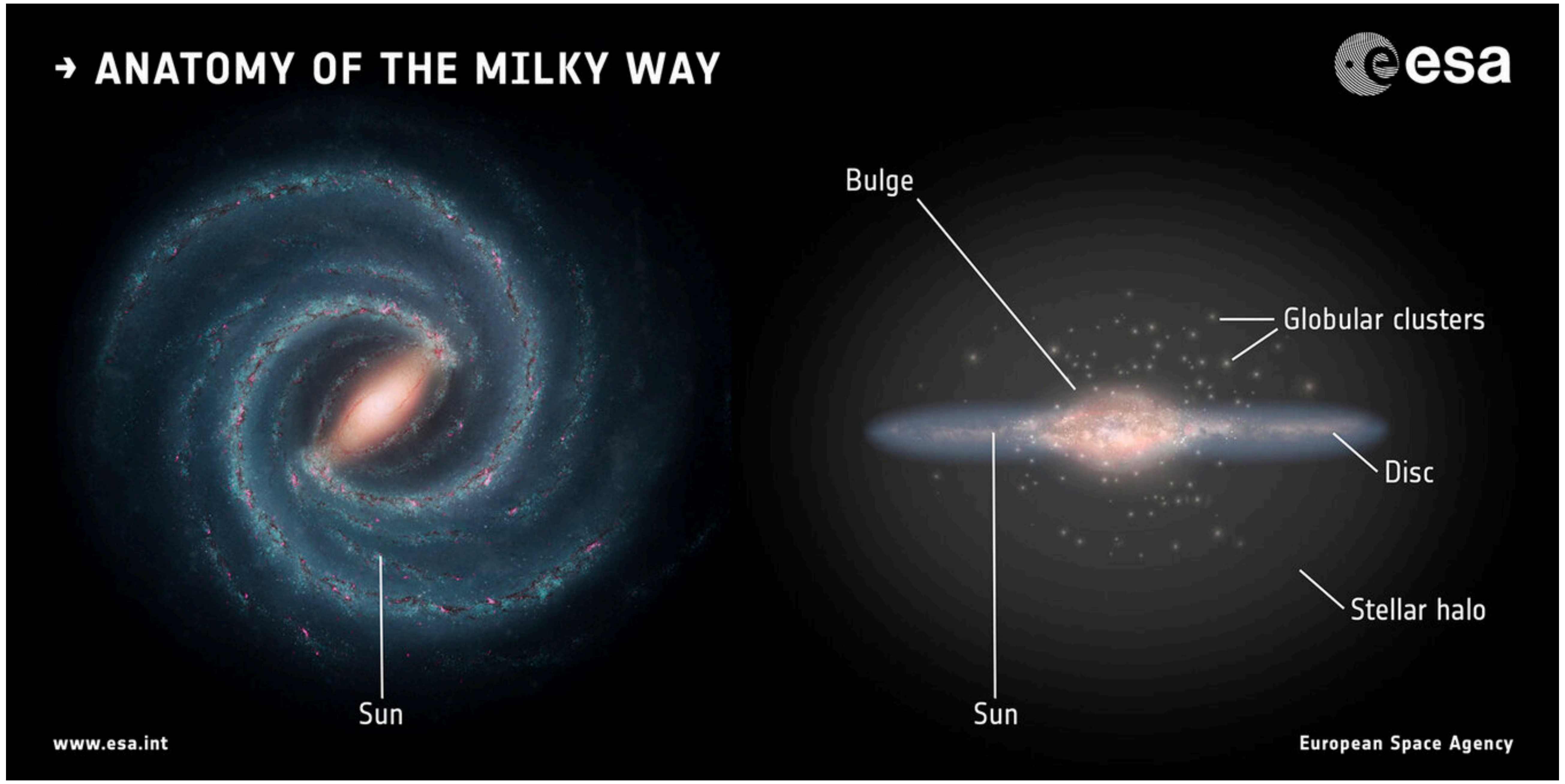
  $$\Phi(r) = -G\frac{M}{r} \qquad r > R,$$

  where $\rho = M/(4\pi R^3/3)$ is the mass density



Galactic potential

- Galactic potential is the collective self-consistent field of all **stars** within the galaxy
  - Determined by the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ that accounts for the mechanical state of the galaxy
- Milky Way has
  - $\sim 10^{11}$ visible stars weighing $\sim 5 \times 10^{10} M_\odot$
  - Gas weighing $\sim 10^{10} M_\odot$
  - Dark matter weighing $\sim 10^{12} M_\odot$
- Gas has little effect on main features of galactic dynamics



- Solar mass $M_\odot = 1.989 \times 10^{33}$ g
- Earth's mass $M_\oplus = 5.976 \times 10^{27}$ g

Scales, units: Motivation

- Let's start with time; What is natural unit of time when discussing the solar system (Kepler motion)
  - 1 s is clearly too small
  - 1 day is reasonable (1 Earth orbit is 365 days)
- What about colliding galaxies
  - Collide over time scale of 1 billion years! 1 day is too small…
  - How do we choose?

Scales, units: Typical values

(1 kpc = 3.085678×10^{21} cm)

- Most of the stars in the galaxy travel on nearly circular orbits in a thin disk whose radius is ~ 10 kpc and thickness ~ 1 kpc
- Typical circular speed of stars is of the order of 200 km/s and the time required to complete a galactic orbit at 10 kpc is about 3×10^8 years

Scales, units: Practical choice

(1 kpc = 3.085678×10^{21} cm)

- Note in cgs units, $G = 6.67 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}$

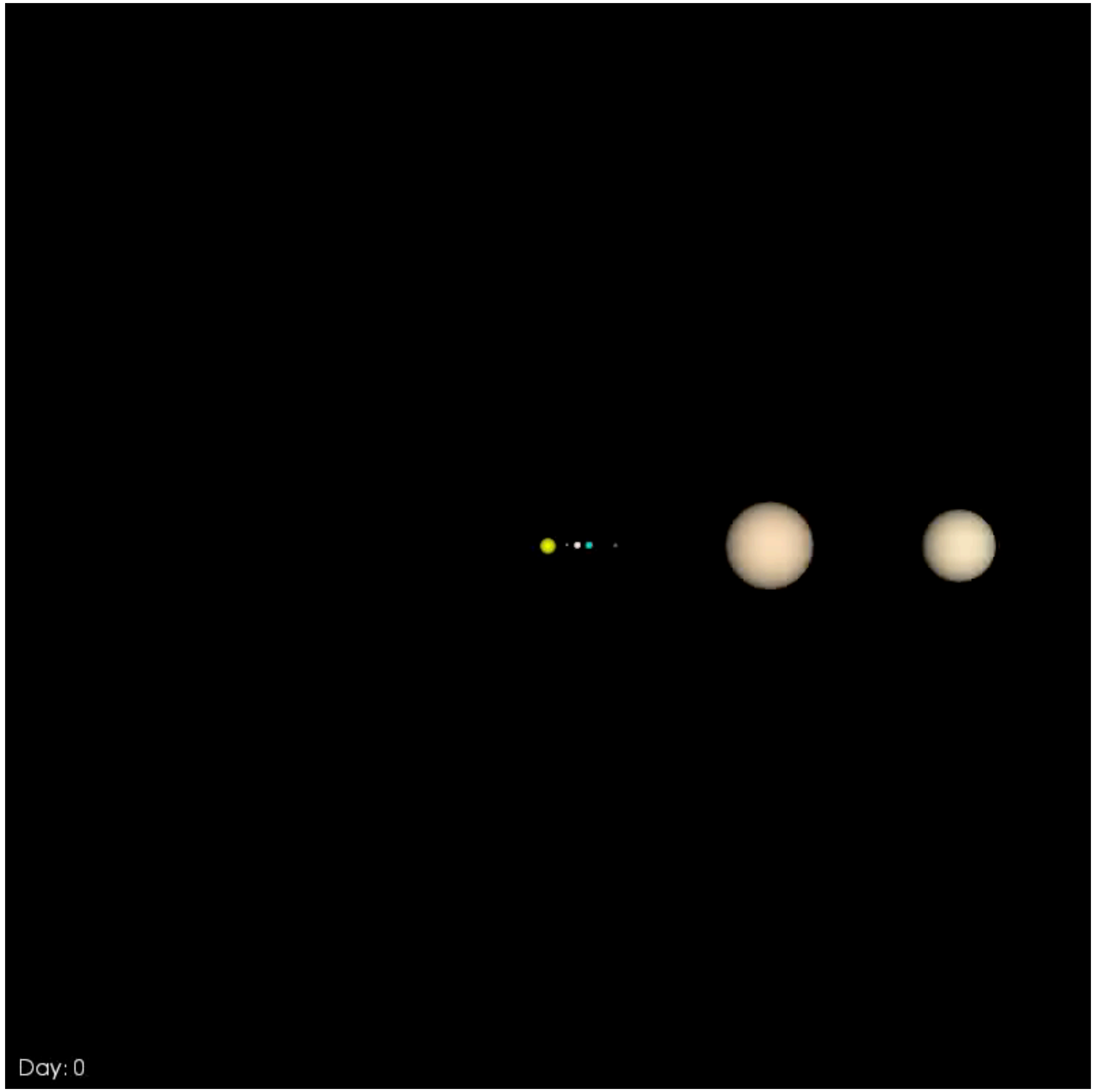
$$[G] = \frac{[L]^3}{[M][T]^2} = \frac{[L][v]^2}{[M]}$$

- It is convenient in numerical simulation to use units such that $G = 1$
  - Normally there are 3 degrees of freedom (e.g. length scale, time scale, mass scale or equivalently, length scale, velocity scale, mass scale, etc.)
  - Setting $G = 1$ removes 1, so we still have two choices.
    - e.g. could choose [L] = 4.5 kpc, [v] = 220 km/s
    - This determines the mass scale as

$$[M] = \frac{[L][v]^2}{[G]} = \frac{(4.5\,\mathrm{kpc})(220\,\mathrm{km/s})^2}{6.67 \times 10^{-8}\,\mathrm{cm^3 g^{-1} s^{-2}}} = 5.06 \times 10^{10}\,M_\odot$$

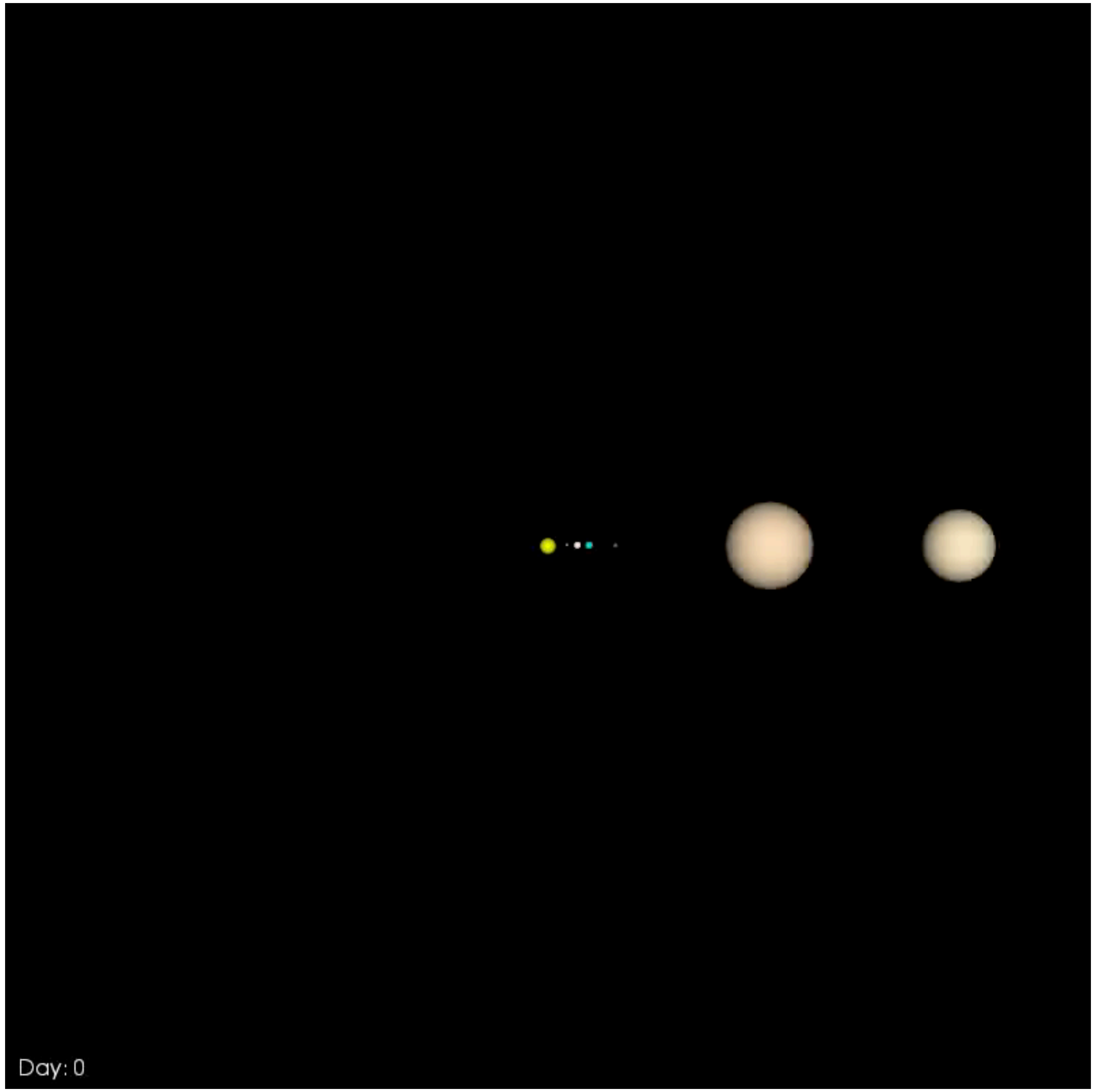
Kepler orbit simulations

- 1 Earth year



Kepler orbit simulations

- 1 Saturn year

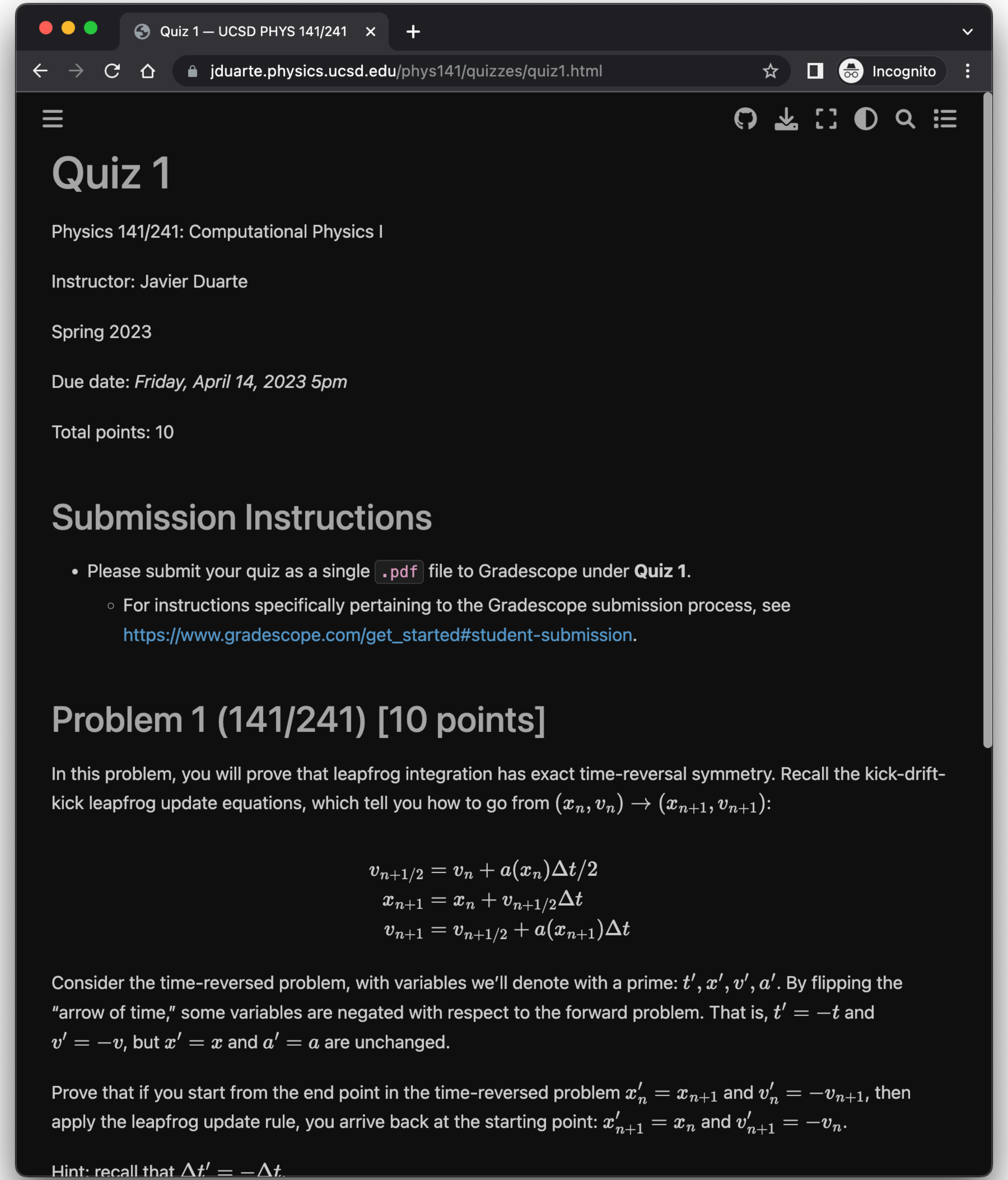


Links

- Astro-Physical Calculator: http://dmaitra.webspace.wheatoncollege.edu/calc.html
- Kepler’s Laws: http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html

Quiz 1

- https://jduarte.physics.ucsd.edu/phys141/quizzes/quiz1.html
- Due Friday 5pm!



Quiz 1

Physics 141/241: Computational Physics I

Instructor: Javier Duarte

Spring 2023

Due date: *Friday, April 14, 2023 5pm*

Total points: 10

Submission Instructions

- Please submit your quiz as a single `.pdf` file to Gradescope under **Quiz 1**.
  - For instructions specifically pertaining to the Gradescope submission process, see https://www.gradescope.com/get_started#student-submission.

Problem 1 (141/241) [10 points]

In this problem, you will prove that leapfrog integration has exact time-reversal symmetry. Recall the kick-drift-kick leapfrog update equations, which tell you how to go from $(x_n, v_n) \to (x_{n+1}, v_{n+1})$:

$$v_{n+1/2} = v_n + a(x_n)\Delta t/2$$
$$x_{n+1} = x_n + v_{n+1/2}\Delta t$$
$$v_{n+1} = v_{n+1/2} + a(x_{n+1})\Delta t$$

Consider the time-reversed problem, with variables we'll denote with a prime: t', x', v', a'. By flipping the "arrow of time," some variables are negated with respect to the forward problem. That is, $t' = -t$ and $v' = -v$, but $x' = x$ and $a' = a$ are unchanged.

Prove that if you start from the end point in the time-reversed problem $x'_n = x_{n+1}$ and $v'_n = -v_{n+1}$, then apply the leapfrog update rule, you arrive back at the starting point: $x'_{n+1} = x_n$ and $v'_{n+1} = -v_n$.

Hint: recall that $\Delta t' = -\Delta t$.