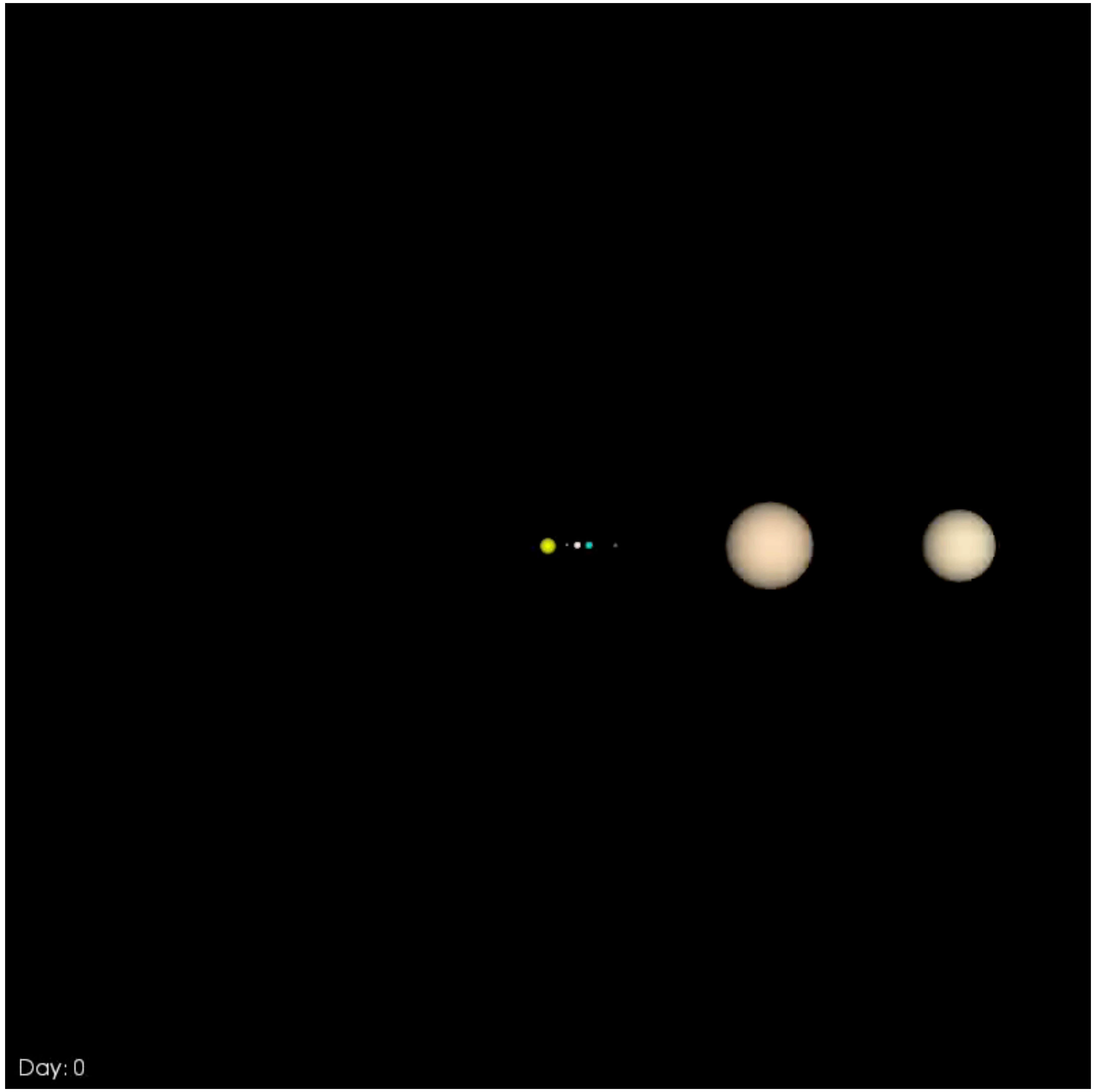
PHYS 141/241

Lecture 06: Gravitational Potential and N-Body Equations (Continued)

Javier Duarte — April 14, 2023

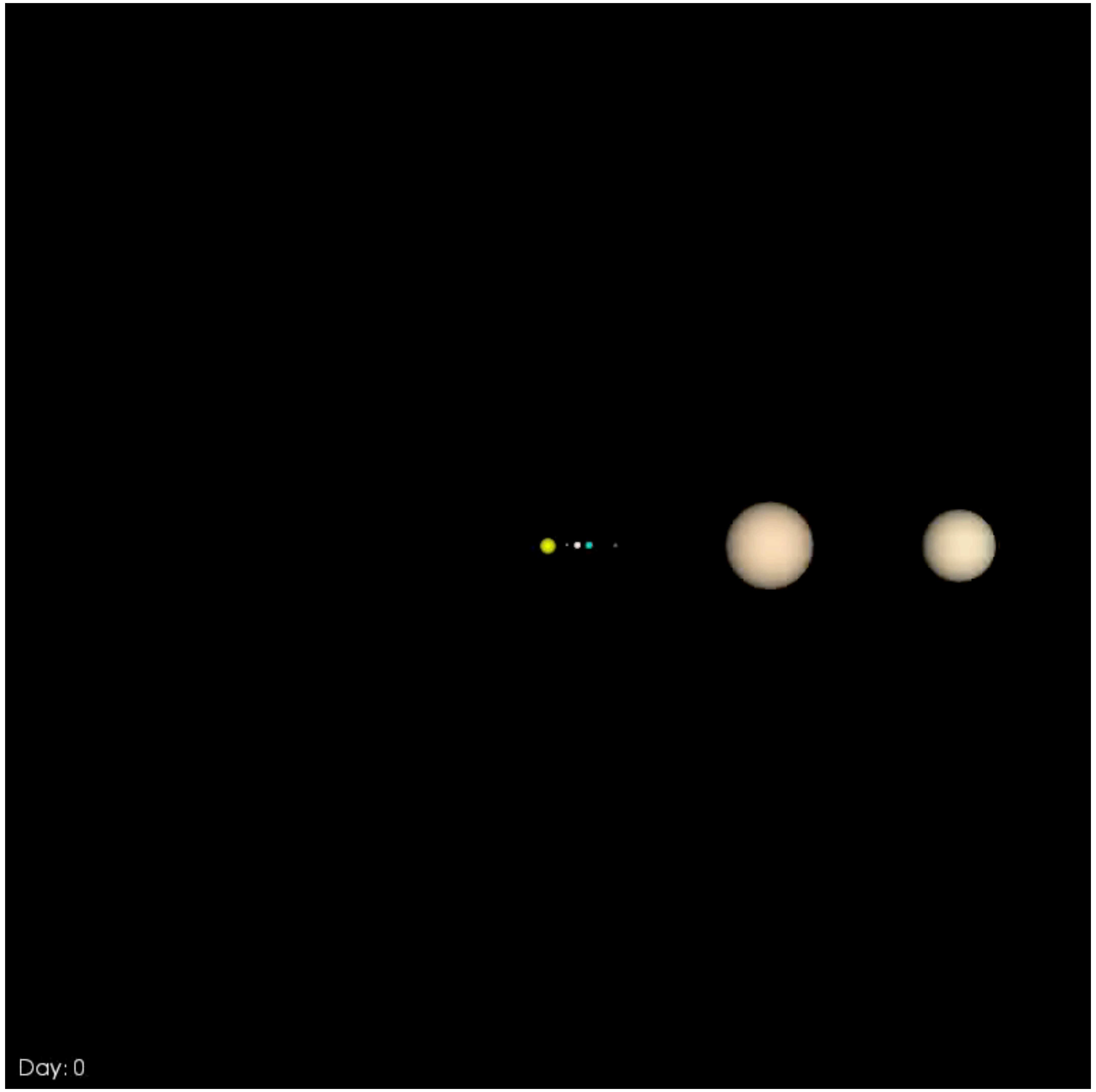
Kepler orbit simulations

- 1 Earth year



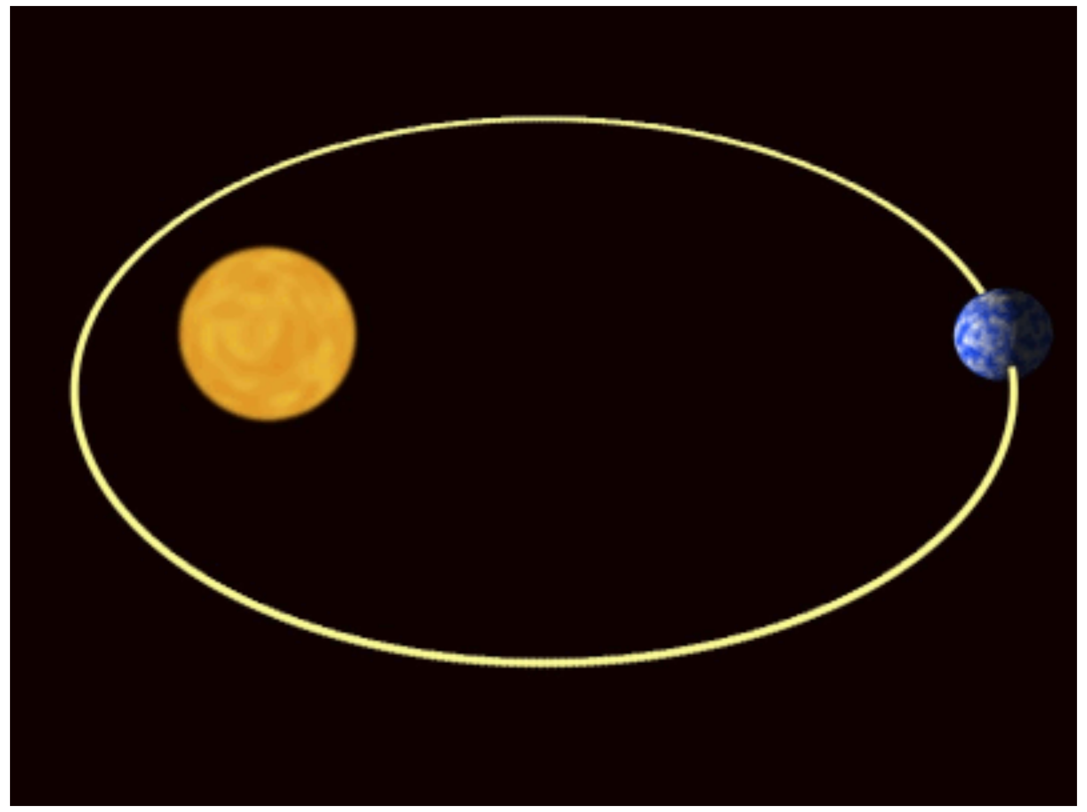
Kepler orbit simulations

- 1 Saturn year



Orbits in Spherical Potentials

- Consider the motion of a star in a spherically-symmetric potential $\Phi(R) = \Phi(|\boldsymbol{r}|)$
- The orbit of the star remains in a plane
  - Why?
    - Because angular momentum is conserved!



- Natural to adopt a polar coordinate system; call the coordinates R and φ
  - $n = 2$ degrees of freedom, so the phase space has 4 dimensions

Lagrangian

- Equations of motion can be derived by starting with the Lagrangian $L = K - U$

$$L(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2} m \left(\dot{R}^2 + R^2 \dot{\varphi}^2 \right) - m\Phi(R)$$

where $\dot{R} = dR/dt$ and $\dot{\varphi} = d\varphi/dt$.

- We will work in units where $m = 1$

$$L(R, \varphi, \dot{R}, \dot{\varphi}) = \frac{1}{2} \left(\dot{R}^2 + R^2 \dot{\varphi}^2 \right) - \Phi(R)$$

Equations of motion

- Differentiating L with respect to $\dot{R}$ and $\dot{\varphi}$ yields the momenta conjugate to R and φ,

$$\frac{\partial L}{\partial \dot{R}} = \dot{R} = v_R$$

$$\frac{\partial L}{\partial \dot{\varphi}} = R^2\dot{\varphi} = Rv_\varphi = J$$

where v_R and v_φ are the velocities in the radial and azimuthal directions

Hamiltonian

- Hamiltonian may be expressed as

$$H(R, \varphi, v_R, J) = \frac{1}{2}(v_R^2 + \frac{J^2}{R^2}) + \Phi(R)$$

- The (1st-order) equations of motions are then

$$dR/dt = \partial H/\partial v_R = v_R$$

$$d\varphi/dt = \partial H/\partial J = J/R^2$$

$$dv_R/dt = -\partial H/\partial R = -\partial \Phi(R)/\partial R + J^2/R^3$$

$$dJ/dt = -\partial H/\partial \varphi = 0$$

- $dJ/dt = 0$ because the conjugate coordinate φ does not appear in H (cyclic coordinate)

Conserved quantities

- The system has two "integrals of motion" (conserved quantities)
  - Total energy E (numerically equal to the value of H)
  - Total angular momentum J

$$E = \frac{1}{2}(v_R^2 + v_\varphi^2) + \Phi(R)$$

$$J = Rv_\varphi$$

- Each of these defines a hypersurface in phase space and the orbit must remain in the intersection of these hypersurfaces
  - Ignoring φ, can visualize surface of constant E and J in 3D space (R, v_R, v_φ)
  - Surfaces of constant E are figures of revolution about R axis, surfaces of constant J are hyperbolas in (R, v_φ) plane (intersection is a closed curve)

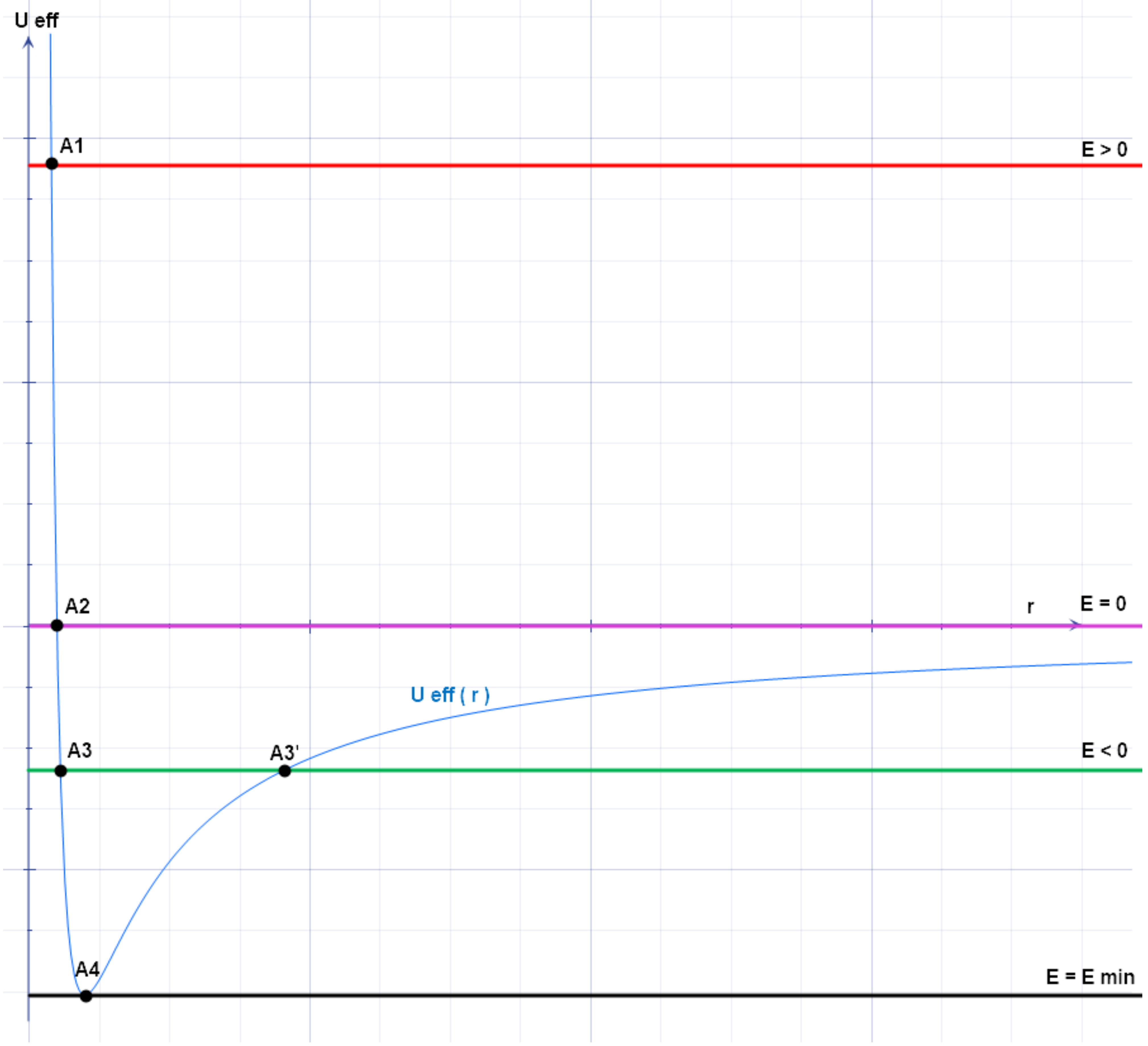
Effective potential

- For an orbit of a given J, the system can be reduced to 1 degree of freedom by defining the *effective potential*,

$$\Psi(R) = \Phi(R) + J^2/(2R^2)$$

- The corresponding equations of motion are then just

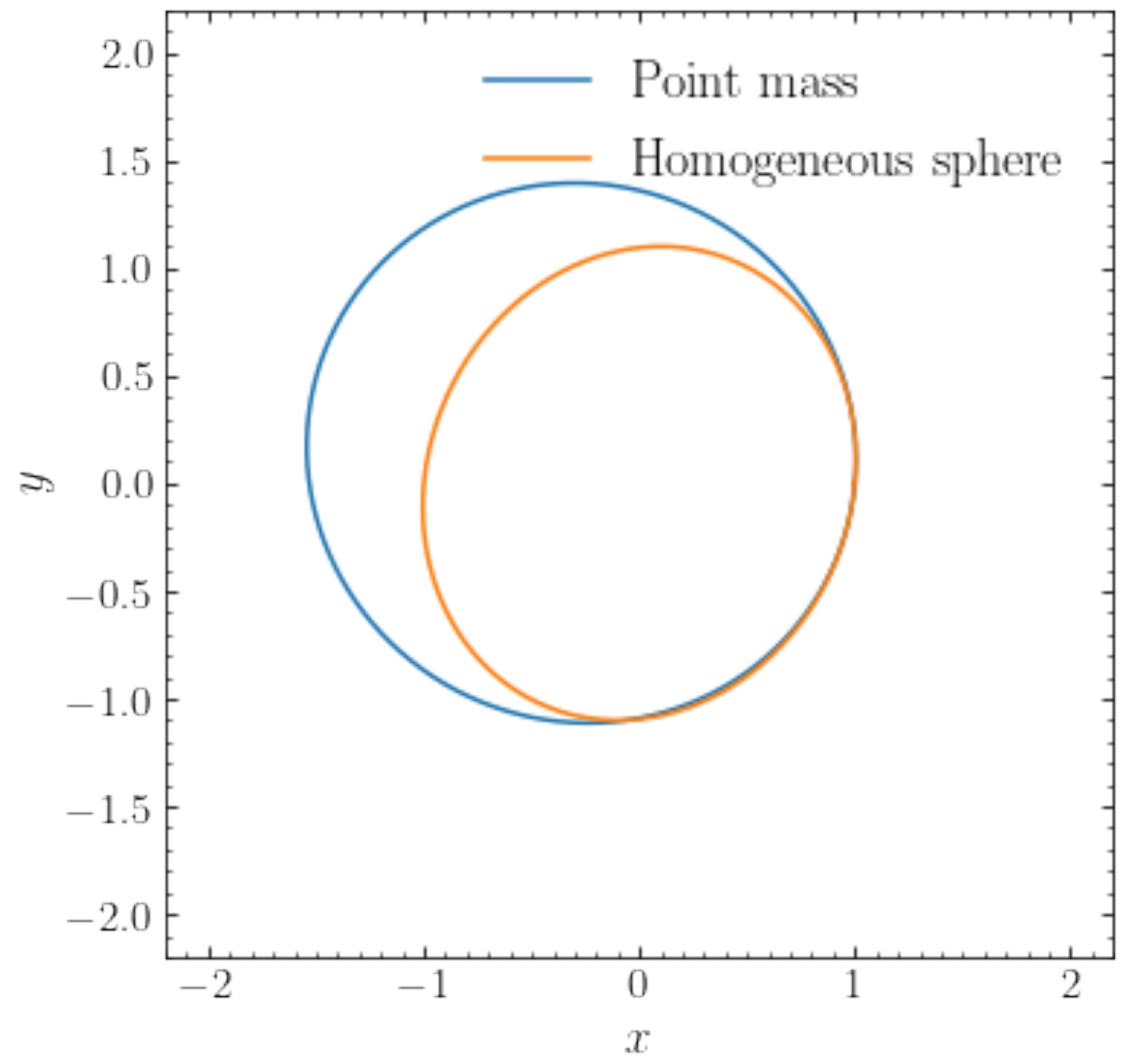
$$dR/dt = v_R$$
$$dv_R/dt = -d\Psi/dR$$

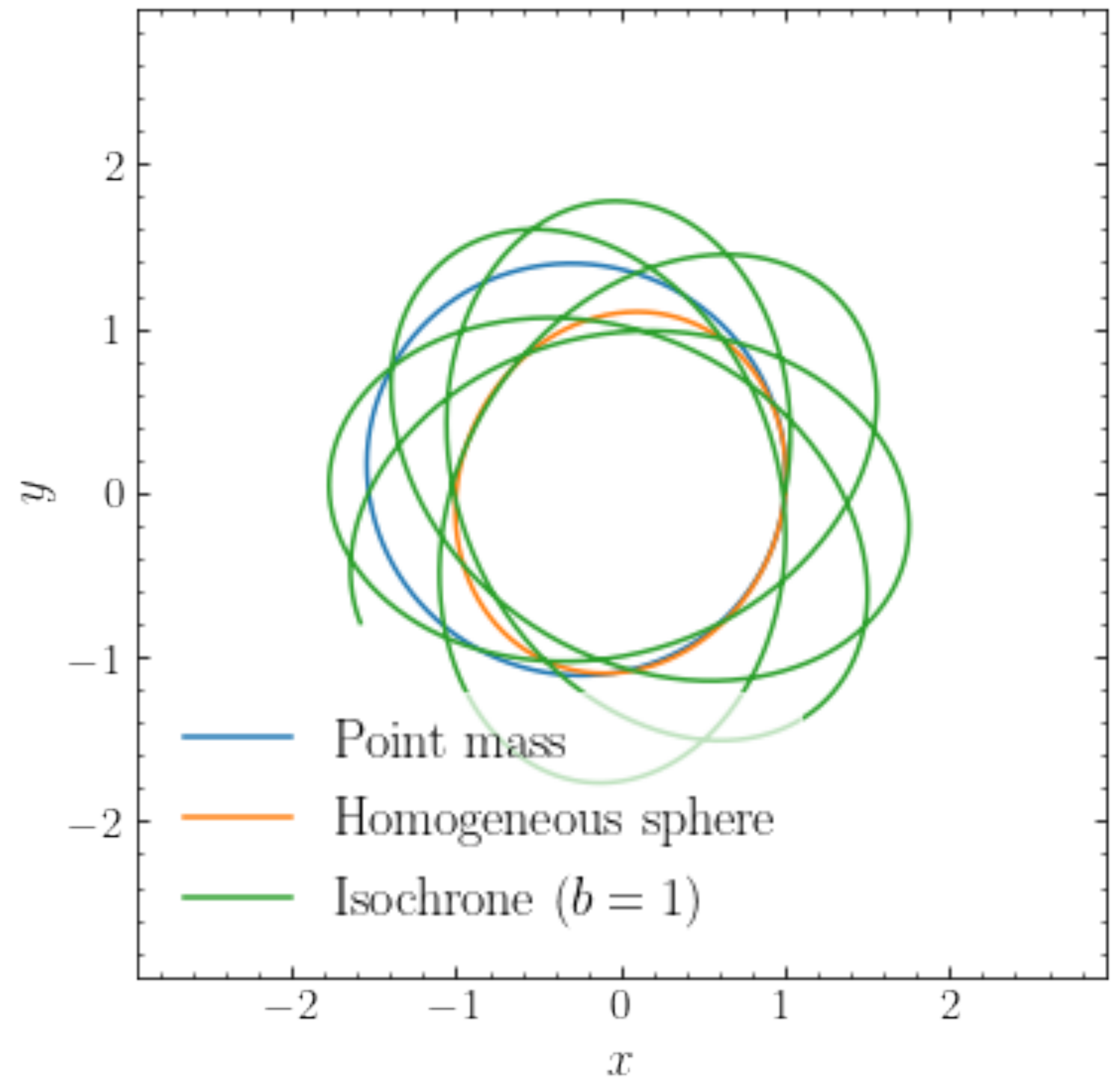


- $\Psi(R)$ diverges as $R \to 0$, so star is prohibited from coming too close to the origin, and shuttles back and forth between turning points $R_{\min}$ and $R_{\max}$

Typical orbits

- In spherical potential, star executes periodic radial motion and periodic azimuthal motion
  - If the two periods are commensurate, orbits close
- Keplerian potential (generated by point mass) is a very special case in which the radial and azimuthal periods are equal ($\Delta\varphi = 2\pi$ between pericenters)
- Harmonic potential (generated by uniform sphere) orbits also close, but radial period is half the azimuthal one ($\Delta\varphi = \pi$ between pericenters)
- In general, most orbits in spherical galaxies are Rosetta advancing by $\pi < \Delta\varphi < 2\pi$ between percenters
  - Figures + Animations: https://galaxiesbook.org/chapters/I-03.-Orbits-in-Spherical-Potentials.html#Orbits-in-the-homogeneous-sphere





N-body equations of motion

- Any system where stellar collisions are rare can be idealized as a a collection of N point-sized bodies, each with mass m_i, position $\boldsymbol{r}_i$, velocity $\boldsymbol{v}_i$
- Hamiltonian is

$$H(\boldsymbol{r}, \boldsymbol{v}) = \frac{1}{2}\sum_{i=2}^{N} m_i \boldsymbol{v}_i^2 - \frac{G}{2}\sum_{i=1}^{N}\sum_{j=1,\ j\neq i}^{N} \frac{m_i m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j}$$

where H depends on all body positions and velocities

  - First sum runs over all N bodies
  - Second runs over all $N(N-1)/2$ pairs of bodies (twice! So factor of $1/2$)
- Equations of motion

$$d\boldsymbol{r}_i/dt = \boldsymbol{v}_i$$

$$d\boldsymbol{v}_i/dt = -G\sum_{j=1,\ j\neq i}^{N} \frac{m_j(\boldsymbol{r}_i - \boldsymbol{r}_j)}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}$$

Conserved quantities

- N-body systems obey several basic conservation laws
- Symmetry of the Hamiltonian is a transformation that leaves the physical system unchanged
- For example, translation in time, $t \to t + \Delta t$ is a symmetry because H is not an explicit function of time
  - $E = K + V = H$ is conserved
- Symmetry with respect to translation in space $\boldsymbol{r}_i \to \boldsymbol{r}_i + \boldsymbol{\Delta r}$ implies conservation of total momentum
- Symmetry with respect to rotation in space gives rise to conservation of total angular momentum

Virial parameters

- For a system in equilibrium

$$2\langle K\rangle + \langle U\rangle = 0$$

- Because $E = K + U$, we have

$$\langle K\rangle = -E, \quad \langle U\rangle = 2E$$

- For an N-body system of total mass M and total energy E, we can define characteristic velocity and length scales

$$V^2 = 2K/M = 2|E|/M$$

$$R = -GM^2/\langle U\rangle = GM^2/(2E)$$

  - Known as *viral* velocity and radius

Timescale

- The quantity

$$t_c = R/V = G\left(\frac{M^5}{8|E|^3}\right)^{1/2} = GM/R^3$$

is an estimate of the time a typical star takes to cross the system

Links

- Astro-Physical Calculator: http://dmaitra.webspace.wheatoncollege.edu/calc.html
- Kepler’s Laws: http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html