PHYS 141/241

Lecture 06: Collisionless Boltzmann Equation

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Quiz 1: Recap

- Quiz 1 solution posted: https://jduarte.physics.ucsd.edu/phys141/quizzes/quiz1_sol.html
  - Note, there may be some typos, let us know!

Virial parameters

- For a system in equilibrium

$$2\langle K\rangle + \langle U\rangle = 0$$

- Because $E = K + U$, we have

$$\langle K\rangle = -E, \quad \langle U\rangle = 2E$$

- For an N-body system of total mass M and total energy E, we can define characteristic velocity and length scales

$$V^2 = 2K/M = 2|E|/M$$

$$R = -GM^2/\langle U\rangle = GM^2/(2E)$$

  - Known as *viral* velocity and radius

Timescale

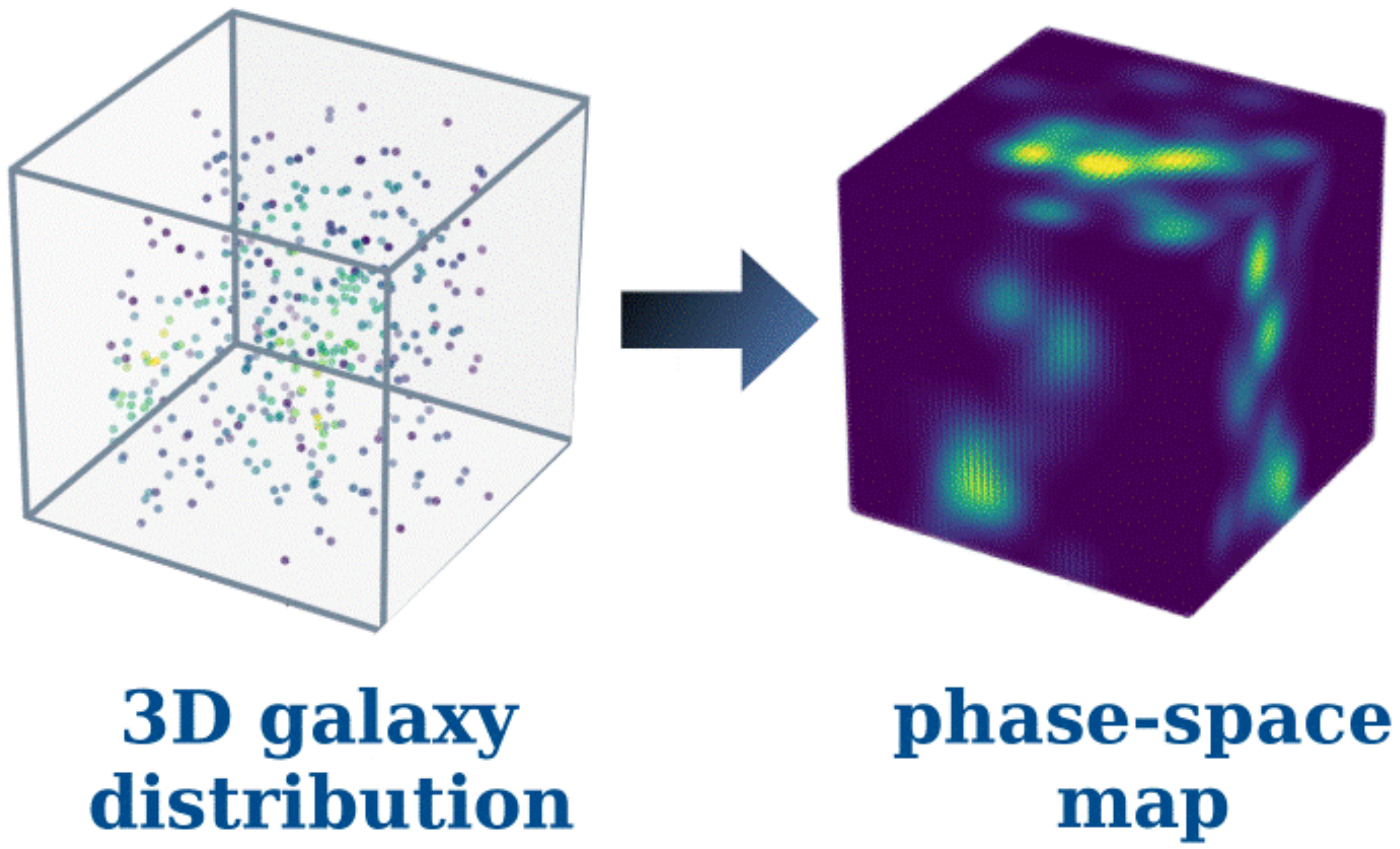
- The quantity

$$t_c = R/V = G\left(\frac{M^5}{8|E|^3}\right)^{1/2} = GM/R^3$$

is an estimate of the time a typical star takes to cross the system

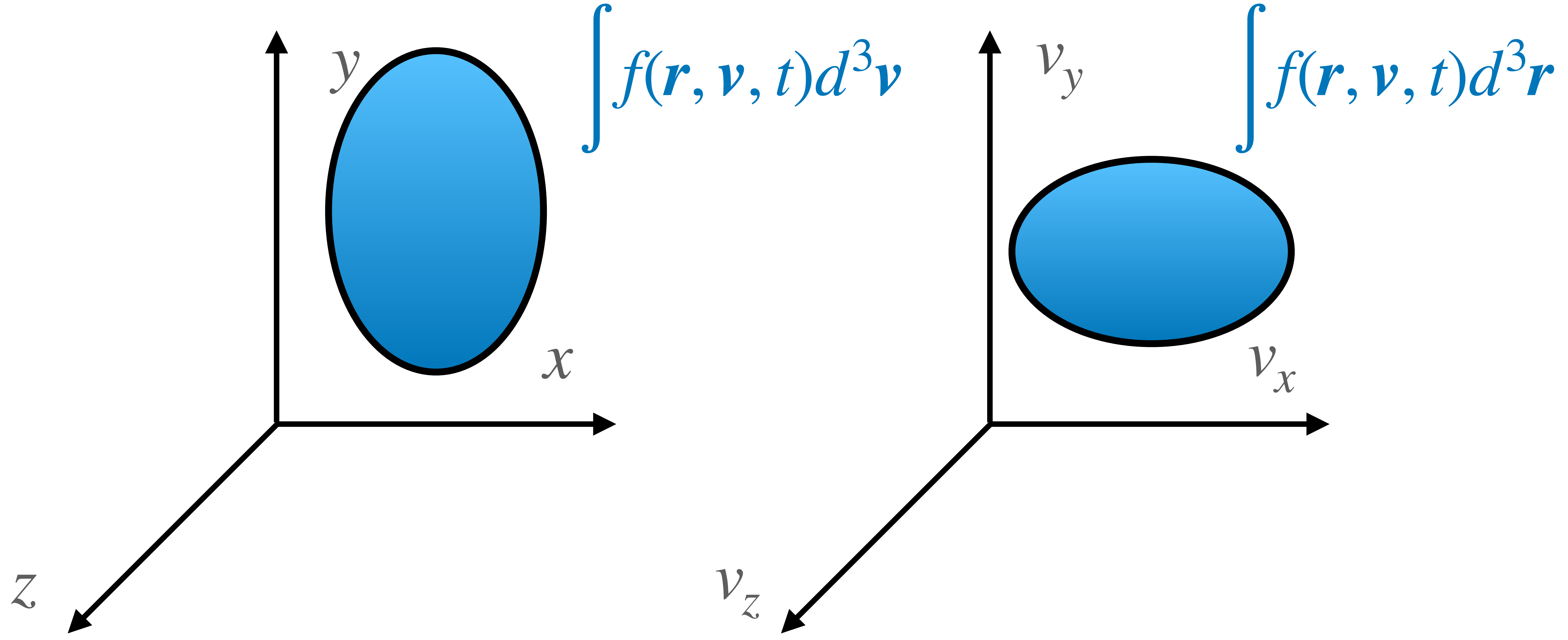
Collisionless dynamics

- A typical galaxy has 10^{11} stars, but only ≤ 100 crossing times old, so the cumulative effects of encounters between stars are not significant
- Idealize galaxy as a continuous mass distribution
  - Each star moves in a smooth gravitational field $\Phi(\boldsymbol{r}, t)$
  - Instead of phase space of $6N$ dimensions, think about motion in a phase space of just 6 dimensions
    - Very important simplification!



Distribution function

- Galaxy may be described by the one-body distribution function
- Let $f(\boldsymbol{r}, \boldsymbol{v}, t)d^3\boldsymbol{r}d^3\boldsymbol{v}$ be the mass of stars in phase space volume $d^3\boldsymbol{r}d^3\boldsymbol{v}$ at $(\boldsymbol{r}, \boldsymbol{v})$ and time t
  - This provides a complete description if stars are uncorrelated (no collisions)

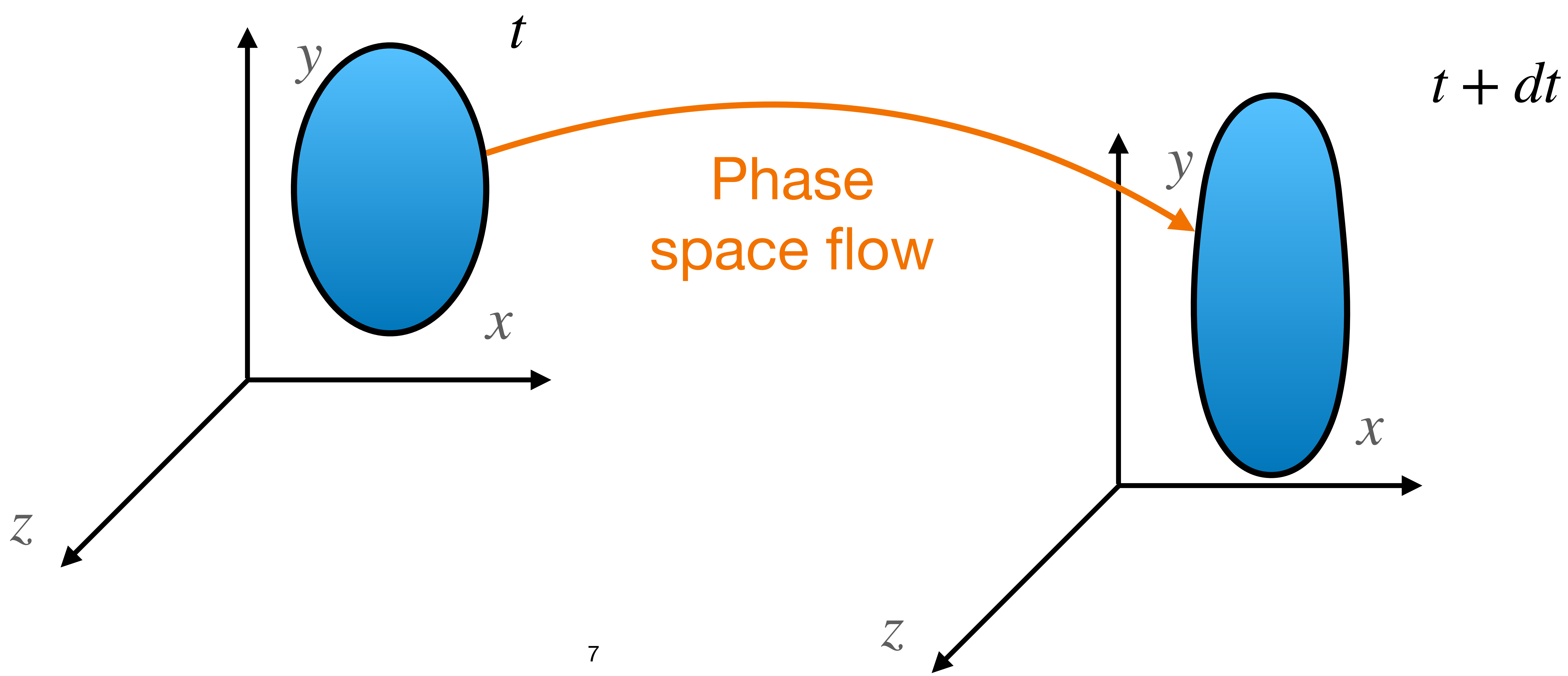


Motion in phase space

- The motion of matter in phase space is governed by the phase space flow

$$(\dot{\boldsymbol{r}}, \dot{\boldsymbol{v}}) = (\boldsymbol{v}, -\nabla\Phi(\boldsymbol{r}, t))$$

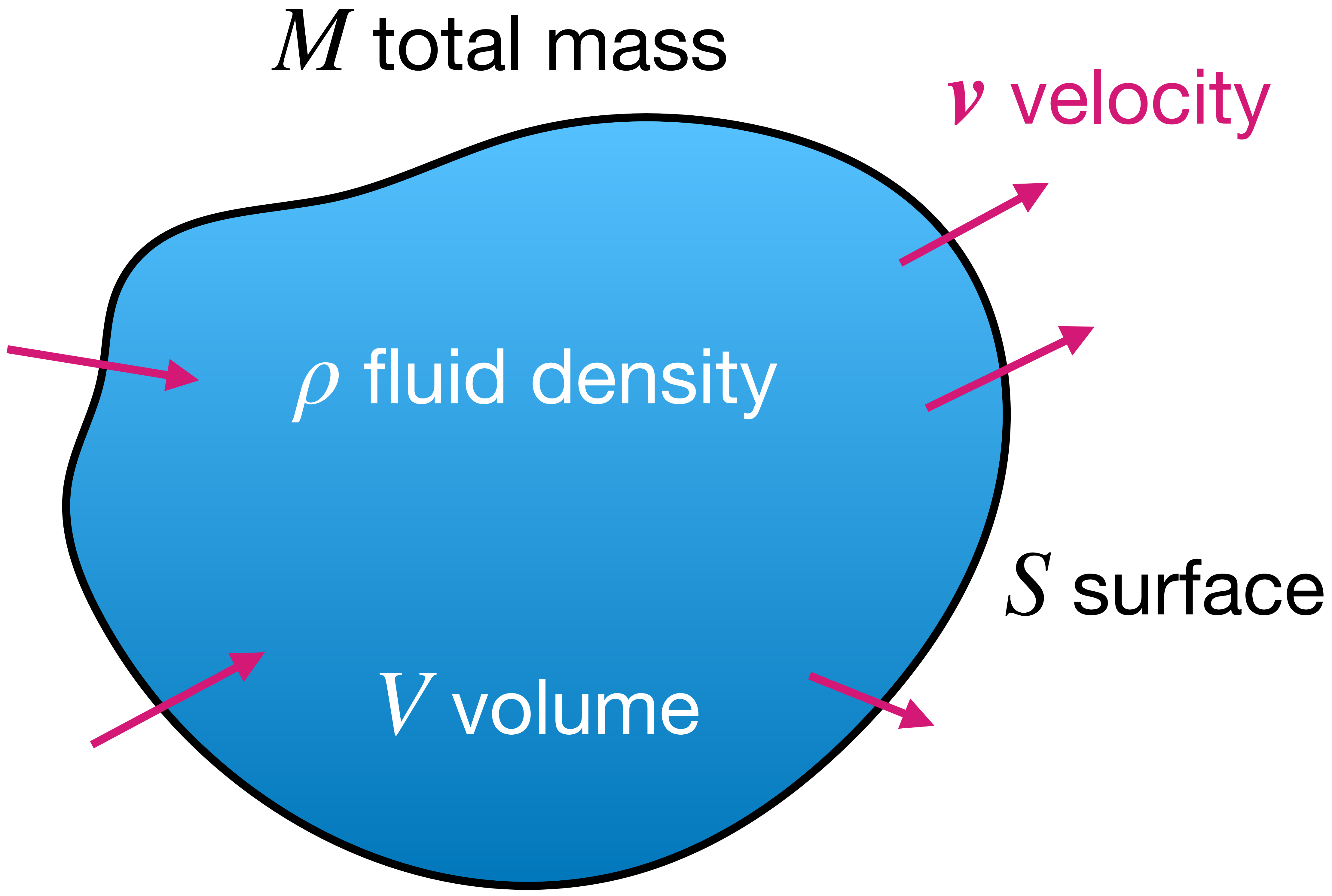
- How does this affect the total mass in $d^3\boldsymbol{r}d^3\boldsymbol{v}$?



Fluid continuity equation

- Fluid continuity equation: rate of change of mass is inflow minus the outflow

$$\frac{dM}{dt} = -\int_S \rho\, \boldsymbol{v} \cdot \hat{\boldsymbol{n}}\ d^2s$$



- Rewriting this in terms of ρ

$$\int_V \frac{d\rho}{dt} d^3x + \int_S \rho\, \boldsymbol{v} \cdot \hat{\boldsymbol{n}}\ d^2s = 0$$

- Local version: $\frac{d\rho}{dt} + \nabla(\rho \boldsymbol{v}) = 0$

- Applied to phase space: $\frac{\partial f}{\partial t} + \nabla_r(f\dot{\boldsymbol{r}}) + \nabla_v(f\dot{\boldsymbol{v}}) = 0$

Collisionless Boltzmann Equation (CBE)

- Combining the phase space flow

$$(\dot{\boldsymbol{r}}, \dot{\boldsymbol{v}}) = (\boldsymbol{v}, -\nabla\Phi(\boldsymbol{r}, t))$$

- And the continuity equation

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{r}}(f\,\dot{\boldsymbol{r}}) + \nabla_{\boldsymbol{v}}(f\,\dot{\boldsymbol{v}}) = 0$$

- The CBE describes the evolution of the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$

$$\frac{\partial f}{\partial t} + (\nabla_{\boldsymbol{r}} f) \cdot v - (\nabla_{\boldsymbol{v}} f) \cdot (\nabla\Phi) = 0$$

Gravitational potential

- Gravitational potential is given self-consistently by Poisson's equation

$$\nabla^2 \Phi(\boldsymbol{r}, t) = 4\pi G \boxed{\int d^3\boldsymbol{v}\, f(\boldsymbol{r}, \boldsymbol{v}, t)}$$

$$\rho(\boldsymbol{r}, t)$$

Conservation of phase space density

- Let $(\boldsymbol{r}(t), \boldsymbol{v}(t))$ be the orbit of a star. What is the rate of change of $f(\boldsymbol{r}, \boldsymbol{v}, t)$ along the star's orbit?

- Let's find the total time derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\nabla_{\boldsymbol{r}} f) \cdot \dot{\boldsymbol{r}} + (\nabla_{\boldsymbol{v}} f) \cdot \dot{\boldsymbol{v}}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\nabla_{\boldsymbol{r}} f) \cdot \boldsymbol{v} - (\nabla_{\boldsymbol{v}} f) \cdot \nabla \Phi = 0$$

  - Zero by the CBE!
  - Phase space density is conserved along every orbit

Jeans Theorem

- Next time…