PHYS 141/241
Lecture 06: Collisionless Boltzmann Equation

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Quiz 1: Recap

• Quiz 1 solution posted: https://jduarte.physics.ucsd.edu/phys141/quizzes/quiz1_sol.html

• Note, there may be some typos, let us know!
Virial parameters

• For a system in equilibrium

\[2\langle K \rangle + \langle U \rangle = 0\]

• Because \( E = K + U \), we have

\[\langle K \rangle = -E, \quad \langle U \rangle = 2E\]

• For an \( N \)-body system of total mass \( M \) and total energy \( E \), we can define characteristic velocity and length scales

\[V^2 = \frac{2K}{M} = 2\frac{|E|}{M}\]

\[R = -\frac{GM^2}{\langle U \rangle} = \frac{GM^2}{2E}\]

• Known as *virial* velocity and radius
Timescale

- The quantity

\[ t_c = \frac{R}{V} = G \left( \frac{M^5}{8 |E|^3} \right)^{1/2} = \frac{GM}{R^3} \]

is an estimate of the time a typical star takes to cross the system.
Collisionless dynamics

• A typical galaxy has $10^{11}$ stars, but only $\leq 100$ crossing times old, so the cumulative effects of encounters between stars are not significant.

• Idealize galaxy as a continuous mass distribution:
  • Each star moves in a smooth gravitational field $\Phi(r, t)$
  • Instead of phase space of $6N$ dimensions, think about motion in a phase space of just 6 dimensions
  • Very important simplification!
Galaxy may be described by the one-body distribution function

Let $f(r, v, t)d^3rd^3v$ be the mass of stars in phase space volume $d^3rd^3v$ at $(r, v)$ and time $t$

- This provides a complete description if stars are uncorrelated (no collisions)
Motion in phase space

• The motion of matter in phase space is governed by the phase space flow

\[(\dot{r}, \dot{v}) = (v, -\nabla \Phi(r, t))\]

• How does this affect the total mass in \(d^3r d^3v\)?
Fluid continuity equation

- Fluid continuity equation: rate of change of mass is inflow minus the outflow

\[
\frac{dM}{dt} = - \int_S \rho \, \mathbf{v} \cdot \mathbf{n} \, d^2s
\]

- Rewriting this in terms of \( \rho \)

\[
\int_V \frac{d\rho}{dt} \, d^3x + \int_S \rho \, \mathbf{v} \cdot \mathbf{n} \, d^2s = 0
\]

  - Local version: \( \frac{d\rho}{dt} + \nabla (\rho \mathbf{v}) = 0 \)

- Applied to phase space: 

\[
\frac{\partial f}{\partial t} + \nabla_r (f \mathbf{r}) + \nabla_v (f \mathbf{v}) = 0
\]
Collisionless Boltzmann Equation (CBE)

• Combining the phase space flow

\[(\dot{r}, \dot{v}) = (v, -\nabla \Phi(r, t))\]

• And the continuity equation

\[
\frac{\partial f}{\partial t} + \nabla_r (f \dot{r}) + \nabla_v (f \dot{v}) = 0
\]

• The CBE describes the evolution of the distribution function \(f(r, v, t)\)

\[
\frac{\partial f}{\partial t} + (\nabla_r f) \cdot v - (\nabla_v f) \cdot (\nabla \Phi) = 0
\]
Gravitational potential

- Gravitational potential is given self-consistently by Poisson’s equation

\[ \nabla^2 \Phi(r, t) = 4\pi G \int d^3v \ f(r, v, t) \]

\[ \rho(r, t) \]
Conservation of phase space density

- Let \( (r(t), v(t)) \) be the orbit of a star. What is the rate of change of \( f(r, v, t) \) along the star’s orbit?

- Let’s find the total time derivative

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + (\nabla_r f) \cdot \dot{r} + (\nabla_v f) \cdot \dot{v}
\]

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + (\nabla_r f) \cdot v - (\nabla_v f) \cdot \nabla \Phi = 0
\]

- Zero by the CBE!

- Phase space density is conserved along every orbit
Jeans Theorem

- Next time…