PHYS 141/241

Lecture 08: Jeans' Theorem and Plummer Model

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Recap

- The CBE describes the evolution of the distribution function $f(\boldsymbol{r}, \boldsymbol{v}, t)$

$$\frac{\partial f}{\partial t} + (\nabla_{\boldsymbol{r}} f) \cdot v - (\nabla_{\boldsymbol{v}} f) \cdot (\nabla \Phi) = 0$$

- Gravitational potential is given self-consistently by Poisson's equation

$$\nabla^2 \Phi(\boldsymbol{r}, t) = 4\pi G \rho(\boldsymbol{r}, t)$$

- $df/dt = 0$ along a star's orbit $(\boldsymbol{r}(t), \boldsymbol{v}(t))$

Jeans' Theorem

- Say $I[\boldsymbol{r}(t), \boldsymbol{v}(t)]$ is an integral of motion that is conserved along any orbit
  $$\frac{d}{dt} I[\boldsymbol{r}(t), \boldsymbol{v}(t)] = 0$$
  - e.g. $E = \frac{1}{2}\boldsymbol{v}^2 + \Phi(\boldsymbol{r})$ and $\boldsymbol{J}$ are integrals of motion
- We can show that $I[\boldsymbol{r}(t), \boldsymbol{v}(t)]$ is a steady state solution of the CBE:

$$\frac{d}{dt} I[\boldsymbol{r}(t), \boldsymbol{v}(t)] = 0 = \nabla_r I \frac{d\boldsymbol{r}}{dt} + \nabla_v I \frac{d\boldsymbol{v}}{dt} = \nabla_r I \boldsymbol{v} - \nabla_v I \cdot \nabla\Phi$$

- Theorem: (1) Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion in the galactic potential, and (2) any function of the integrals yields a steady-state solution of the CBE.

Jeans' Theorem

- Reiterating (2), any function

  $$F(r, v) = F(I_1(r, v), I_2(r, v), \ldots)$$

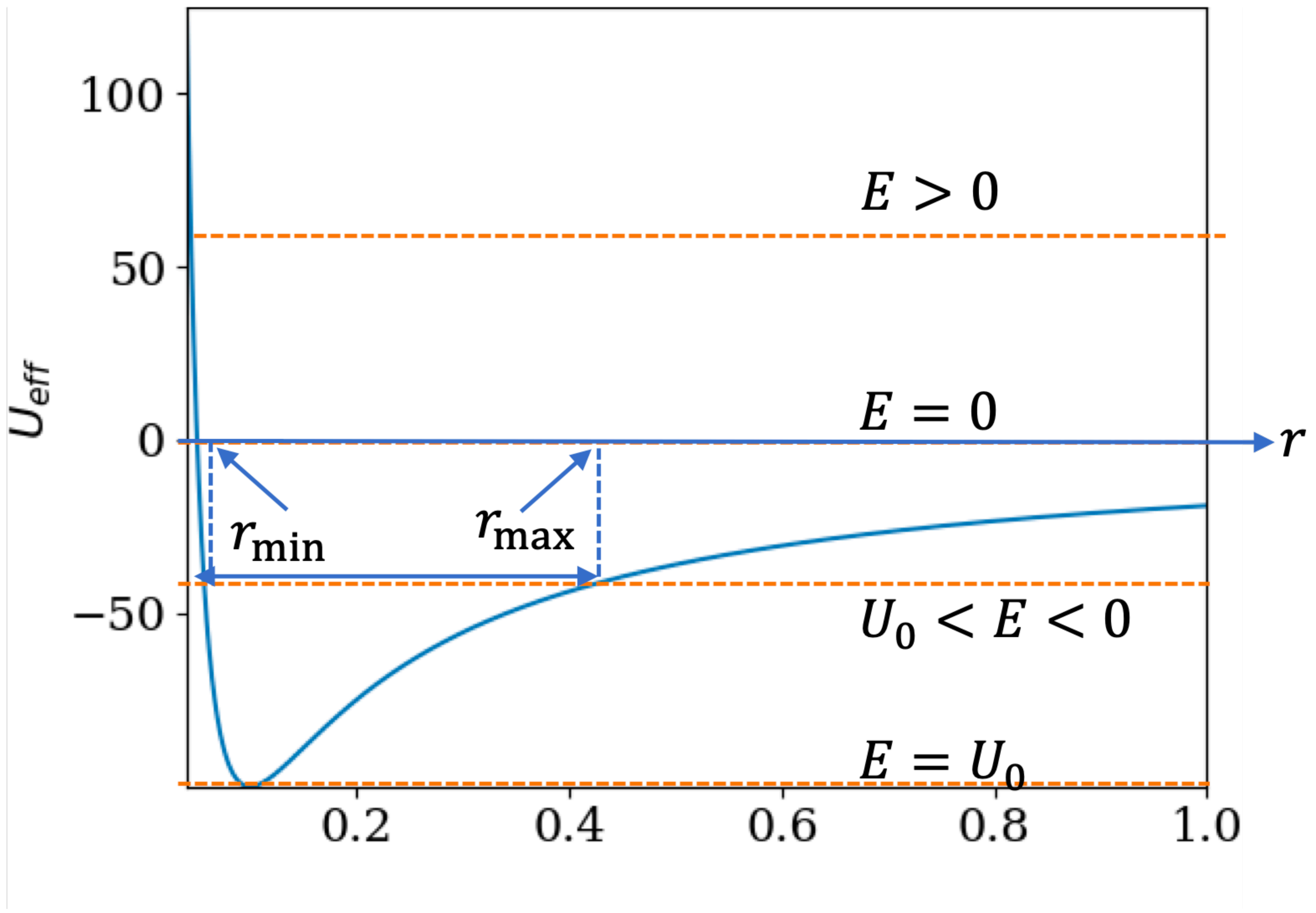
  is guaranteed to be a solution of the CBE

- We can use this fact to construct equilibrium models of stellar systems, e.g. Plummer model

Isotropic models

- Simplest use of Jeans' theorem is the construction of isotropic models of spherical galaxies

- Distribution function $f(\boldsymbol{r}, \boldsymbol{v}) = f(E)$ is only a function of the energy
- Self-consistent Poisson equation in spherical coordinates



$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\rho(r)$$

- Typical to set boundary condition $\Phi \to 0$ as $r \to \infty$
  - Escape energy is 0 and stars at radius r have energies $\Phi(r) < E < 0$

Isotropic models: From f to ρ

- Mass density is integral of f over all velocities; velocity distribution is isotropic so
$$\rho = \int d^3\boldsymbol{v} f(\boldsymbol{r}, \boldsymbol{v}) = 4\pi \int_0^{v_e} dv\ v^2 f\left(\frac{1}{2}v^2 + \Phi(r)\right)$$
  - Where escape velocity is $v_e = \sqrt{-2\Phi(r)}$

Plummer Model

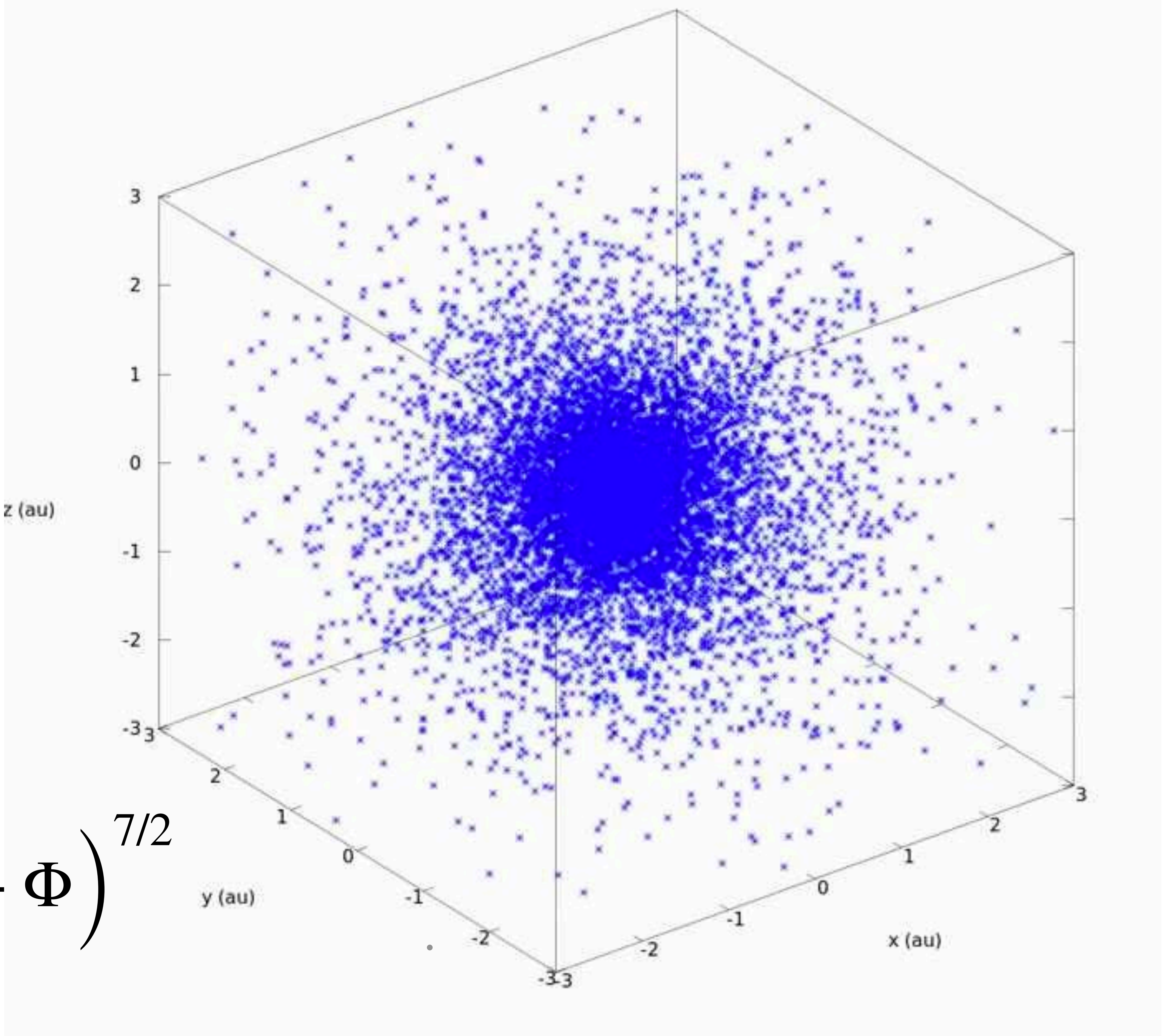
- Choose

$$f = \begin{cases} F(-E)^{7/2} & \text{if } E < 0 \\ 0 & \text{if } E \geq 0 \end{cases}$$

- What is ρ for this model?

$$\rho = 4\pi \int_0^{v_e} dv\ v^2 f\left(\frac{1}{2}v^2 + \Phi\right)$$

$$= 4\pi F \int_0^{\sqrt{-2\Phi}} dv\ v^2 \left(-\frac{1}{2}v^2 - \Phi\right)^{7/2}$$



Plummer Model

- First we can relate ρ and Φ
- If we make the substitution $v^2 = -2\Phi\cos^2\theta,\ dv = \sqrt{-2\Phi}\sin\theta d\theta$

Important: ρ rises as 5th power of $-\Phi$ when $-\Phi > 0$ and is zero otherwise

$$\rho = 4\pi F \int_0^{\sqrt{-2\Phi}} dv\ v^2 \left(-\frac{1}{2}v^2 - \Phi\right)^{7/2}$$

$$= 4\pi F \int_{\pi/2}^{0} (-2\Phi)^{1/2} \sin\theta d\theta (-2\Phi\cos^2\theta) \left(\Phi\cos^2\theta - \Phi\right)^{7/2}$$

$$= 2^{7/2}\pi F (-\Phi)^5 \int_0^{\pi/2} d\theta \sin\theta \cos^2\theta \ \left(1 - \cos^2\theta\right)^{7/2} = \boxed{c_p(-\Phi)^5}$$

Solving for Potential

- Now we can solve for $\Phi(r)$ from Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G c_p(-\Phi)^5$$

a is called the Plummer radius

- Defining $a = ((4/3)\pi G c_p \Phi_0^4)^{-1/2}$ and $\Phi_0 = \Phi(0)$ is useful

- Solution is given by

$$\Phi(r) = \frac{\Phi_0}{\sqrt{1 + r^2/a^2}}$$

Corresponding density

- Thus, the Plummer density is

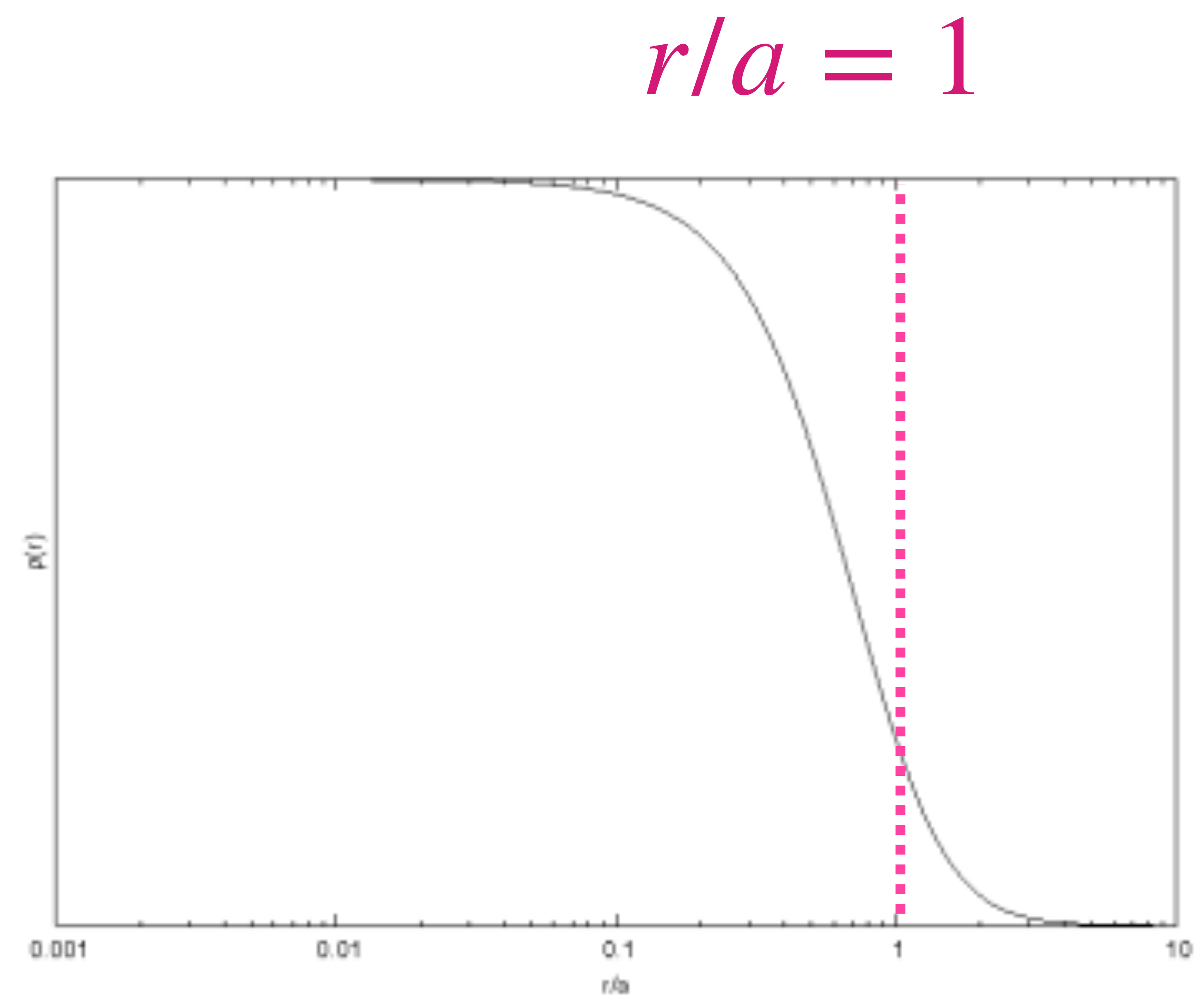
$$\rho(r) = c_p(-\Phi)^5 = c_p \Phi_0^5 \left(1 + \frac{r^2}{a^2}\right)^{-5/2}$$

- Nonzero density everywhere! Total mass is finite: $M = -\frac{\Phi_0 a}{G}$

- In terms of the total mass:

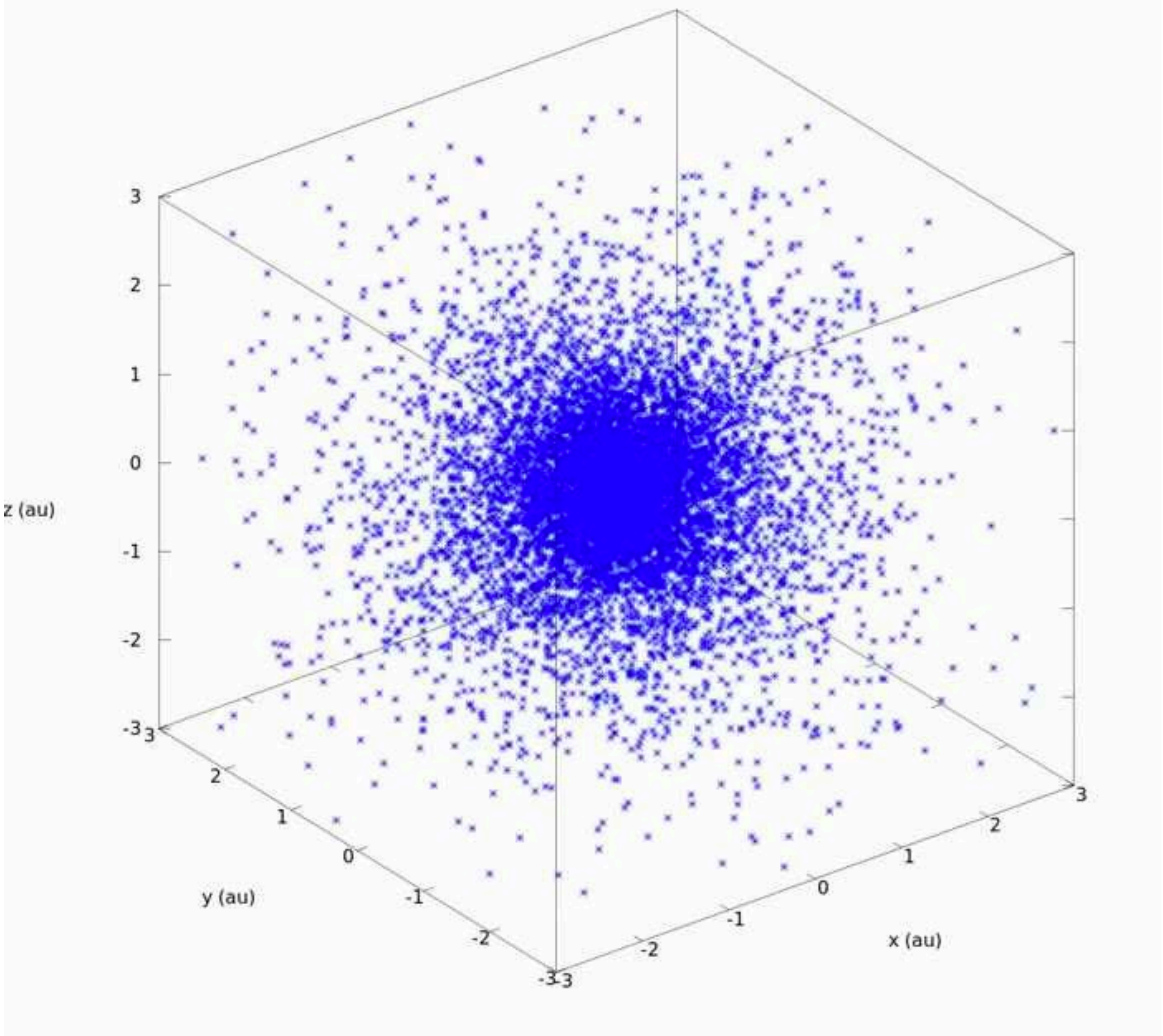
$$\rho(r) = \frac{3M}{4\pi a^3} \left(1 + \frac{r^2}{a^2}\right)^{-5/2}$$

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}}$$



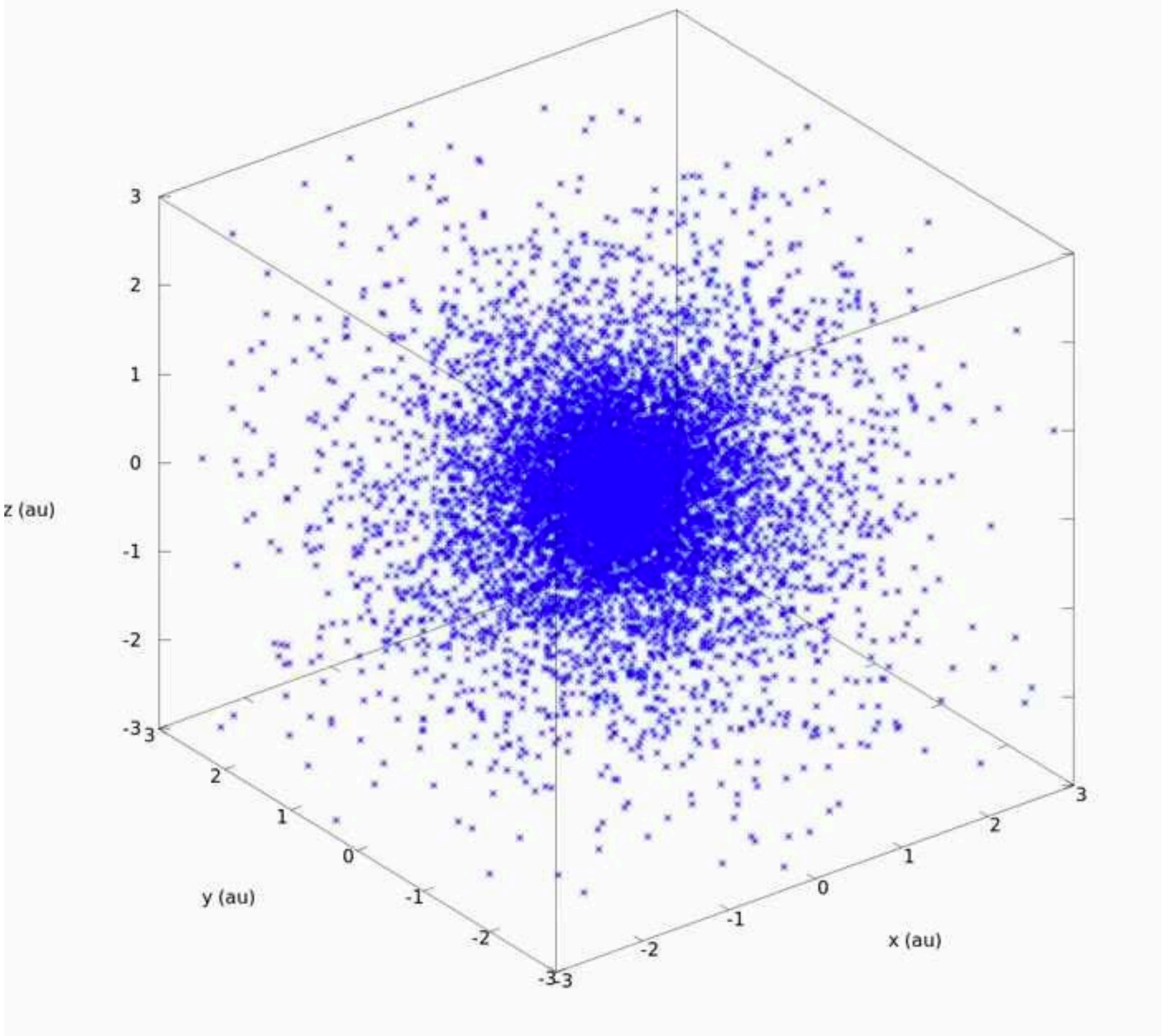
Realism?

- This model was originally devised to describe observations of star clusters
- Actually not a very good model for elliptical galaxies
- Most of the mass lies within a nearly-constant-density core, and at large r the density falls as r^5
    - Steeper than the density profiles of elliptical galaxies



Stationary solution

- But it is still quite useful!
- In isolation, it is "stationary,"
- That is, we can evolve it forward in time and while points move around, the distribution is the same



Family of solutions

Name	$\rho(r)$	$\Phi(r)$
Plummer	$\frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}$	$\frac{-GM}{\sqrt{r^2+a^2}}$
Hernquist	$\frac{M}{2\pi}\frac{a}{r(r+a)^3}$	$\frac{-GM}{r+a}$
Jaffe	$\frac{M}{4\pi}\frac{a}{r^2(r+a)^2}$	$\frac{GM}{a}\ln\left(\frac{a}{r+a}\right)$
Gamma	$\frac{(3-\gamma)M}{4\pi a^3}\frac{a^4}{r^\gamma(r+a)^{4-\gamma}}$	$\frac{GM}{a}\begin{cases}\frac{1}{\gamma-2}\left[1-\left(\frac{r}{r+a}\right)^{2-\gamma}\right], & \gamma\neq 2\\ \ln\left(\frac{r}{r+a}\right), & \gamma=2\end{cases}$

“Hypervirial”

- https://arxiv.org/abs/astro-ph/0501091

  Distribution functions

  $f \propto L^{p-2} E^{(3p+1)/2}$