PHYS 141/241
Lecture 08: Jeans’ Theorem and Plummer Model

Javier Duarte — April 19, 2023
Recap

- The CBE describes the evolution of the distribution function $f(r, v, t)$

$$\frac{\partial f}{\partial t} + (\nabla_r f) \cdot v - (\nabla_v f) \cdot (\nabla \Phi) = 0$$

- Gravitational potential is given self-consistently by Poisson’s equation

$$\nabla^2 \Phi(r, t) = 4\pi G \rho(r, t)$$

- $df/dt = 0$ along a star’s orbit $(r(t), v(t))$
Jean’s Theorem

- Say \( I[r(t), v(t)] \) is an integral of motion that is conserved along any orbit
  \[
  \frac{d}{dt} I[r(t), v(t)] = 0
  \]
  - e.g. \( E = \frac{1}{2}v^2 + \Phi(r) \) and \( J \) are integrals of motion
- We can show that \( I[r(t), v(t)] \) is a steady state solution of the CBE:
  \[
  \frac{d}{dt} I[r(t), v(t)] = 0 = \nabla_r I \frac{dr}{dt} + \nabla_v I \frac{dv}{dt} = \nabla_r I v - \nabla_v I \cdot \nabla \Phi
  \]
- Theorem: (1) Any steady-state solution of the CBE depends on the phase-space coordinates only through integrals of motion in the galactic potential, and (2) any function of the integrals yields a steady-state solution of the CBE.
Jeans’ Theorem

• Reiterating (2), any function

\[ F(r, v) = F(I_1(r, v), I_2(r, v), \ldots) \]

is guaranteed to be a solution of the CBE

• We can use this fact to construct equilibrium models of stellar systems, e.g. Plummer model
Isotropic models

- Simplest use of Jeans’ theorem is the construction of isotropic models of spherical galaxies
- Distribution function $f(r, v) = f(E)$ is only a function of the energy
- Self-consistent Poisson equation in spherical coordinates

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r)$$

- Typical to set boundary condition $\Phi \to 0$ as $r \to \infty$
  - Escape energy is 0 and stars at radius $r$ have energies $\Phi(r) < E < 0$
Isotropic models: From $f$ to $\rho$

- Mass density is integral of $f$ over all velocities; velocity distribution is isotropic so

$$\rho = \int d^3vf(r, v) = 4\pi \int_0^{v_e} dv \ v^2f\left(\frac{1}{2}v^2 + \Phi(r)\right)$$

- Where escape velocity is $v_e = \sqrt{-2\Phi(r)}$
Plummer Model

- Choose
  \[ f = \begin{cases} 
  F(-E)^{7/2} & \text{if } E < 0 \\
  0 & \text{if } E \geq 0 
  \end{cases} \]

- What is \( \rho \) for this model?

  \[
  \rho = 4\pi \int_0^{v_e} dv \ v^2 f \left( \frac{1}{2} v^2 + \Phi \right)
  \]

  \[
  = 4\pi F \int_0^{\sqrt{-2\Phi}} dv \ v^2 \left( -\frac{1}{2} v^2 - \Phi \right)^{7/2}
  \]
Plummer Model

• First we can relate $\rho$ and $\Phi$

• If we make the substitution $v^2 = -2\Phi \cos^2 \theta$, $dv = \sqrt{-2\Phi} \sin \theta d\theta$

$$\rho = 4\pi F \int_0^{\sqrt{-2\Phi}} dv \ v^2 \left(-\frac{1}{2}v^2 - \Phi\right)^{7/2}$$

$$= 4\pi F \int_0^{\pi/2} (-2\Phi)^{1/2} \sin \theta d\theta (-2\Phi \cos^2 \theta) \left(\Phi \cos^2 \theta - \Phi\right)^{7/2}$$

$$= 2^{7/2} \pi F (-\Phi)^5 \int_0^{\pi/2} d\theta \sin \theta \cos^2 \theta \left(1 - \cos^2 \theta\right)^{7/2} = c_p (-\Phi)^5$$

Important: $\rho$ rises as 5th power of $-\Phi$ when $-\Phi > 0$ and is zero otherwise
Solving for Potential

1. Now we can solve for \( \Phi(r) \) from Poisson’s equation

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G c_p (-\Phi)^5
\]

2. Defining \( a = \left( \frac{4}{3} \pi G c_p \Phi_0^4 \right)^{-1/2} \) and \( \Phi_0 = \Phi(0) \) is useful

3. Solution is given by

\[
\Phi(r) = \frac{\Phi_0}{\sqrt{1 + r^2/a^2}}
\]
Corresponding density

- Thus, the Plummer density is

\[ \rho(r) = c_p (-\Phi)^5 = c_p \Phi_0^5 \left(1 + \frac{r^2}{a^2}\right)^{-5/2} \]

- Nonzero density everywhere! Total mass is finite: \( M = -\frac{\Phi_0 a}{G} \)

- In terms of the total mass:

\[ \rho(r) = \frac{3M}{4\pi a^3} \left(1 + \frac{r^2}{a^2}\right)^{-5/2} \]

\[ \Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}} \]
Realism?

- This model was originally devised to describe observations of star clusters.
- Actually not a very good model for elliptical galaxies.
- Most of the mass lies within a nearly-constant-density core, and at large $r$ the density falls as $r^5$.
- Steeper than the density profiles of elliptical galaxies.
Stationary solution

• But it is still quite useful!
• In isolation, it is “stationary,”
• That is, we can evolve it forward in time and while points move around, the distribution is the same
## Family of solutions

<table>
<thead>
<tr>
<th>Name</th>
<th>( \rho(r) )</th>
<th>( \Phi(r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plummer</td>
<td>( \frac{3M}{4\pi a^3} \left(1 + \frac{r^2}{a^2}\right)^{-5/2} )</td>
<td>( -\frac{GM}{\sqrt{r^2 + a^2}} )</td>
</tr>
<tr>
<td>Hernquist</td>
<td>( \frac{M}{2\pi} \frac{r(r + a)^3}{a} )</td>
<td>( -\frac{GM}{r + a} )</td>
</tr>
<tr>
<td>Jaffe</td>
<td>( \frac{M}{4\pi} \frac{r^2(r + a)^2}{a} )</td>
<td>( \frac{GM}{a} \ln \left( \frac{a}{r + a} \right) )</td>
</tr>
</tbody>
</table>
| Gamma    | \( \frac{(3 - \gamma)M}{4\pi a^3} \frac{a^4}{r^\gamma(r + a)^{4-\gamma}} \) | \( \frac{GM}{a} \left\{ \frac{1}{\gamma-2} \left[ 1 - \left( \frac{r}{r+a} \right)^{2-\gamma} \right] \right. \), \( \gamma \neq 2 \)  
|          |                                                          | \left. \ln \left( \frac{r}{r+a} \right) \right. \), \( \gamma = 2 \) |
“Hypervirial”


Distribution functions

\[ f \propto L^{p-2} E^{(3p+1)/2} \]