PHYS 141/241

Lecture 09: Plummer Model and Monte Carlo

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Recap: Plummer Model

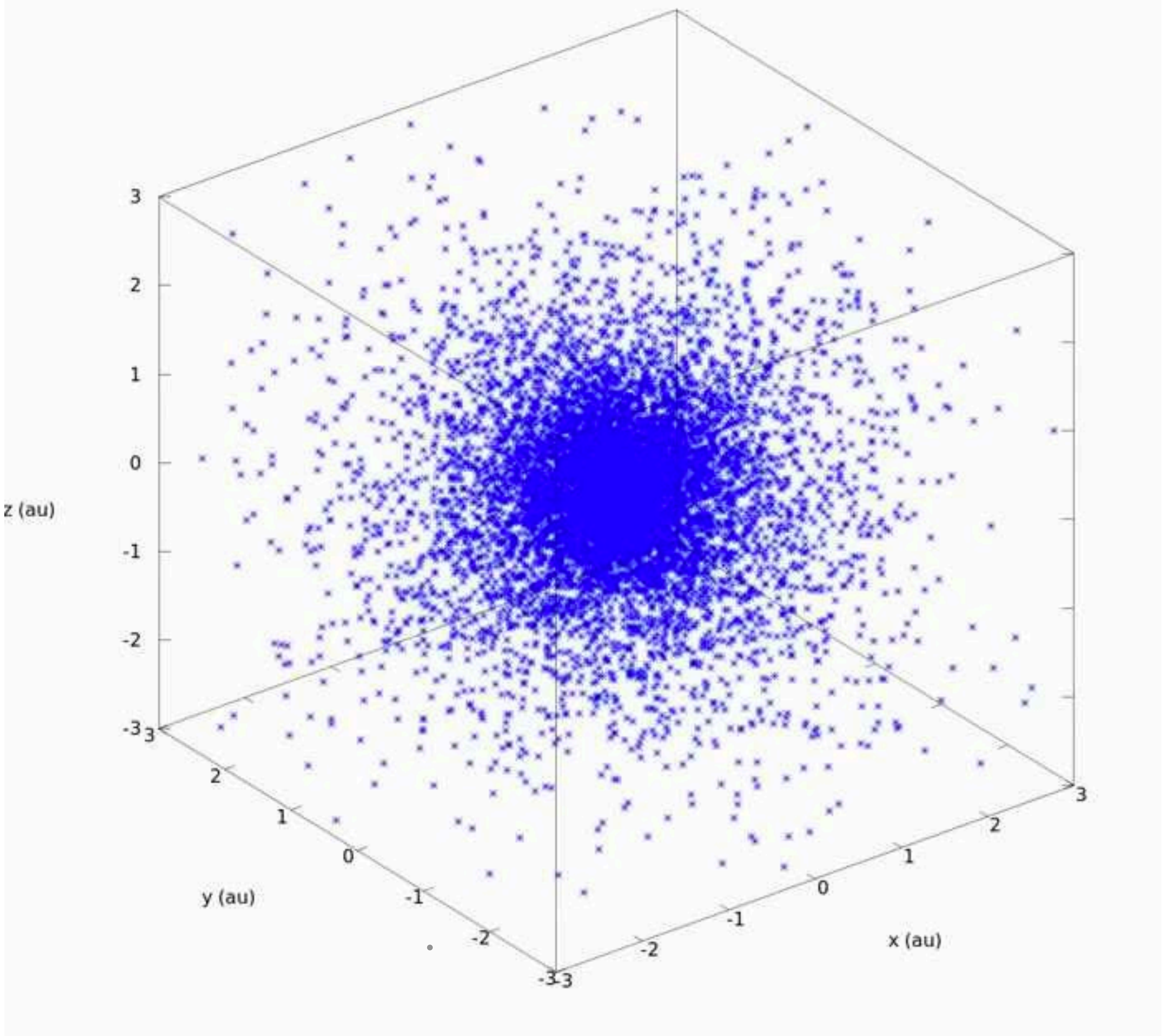
- Distribution function

$$f = \begin{cases} F(-E)^{7/2} & \text{if } E < 0 \\ 0 & \text{if } E \geq 0 \end{cases}$$

- We found

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2}$$

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}}$$



Quiz 2

- W also found $\rho = c_p(-\Phi)^5$ where

$$c_p = 2^{7/2}\pi F \int_0^{\pi/2} d\theta \sin\theta \cos^2\theta \ \left(1 - \cos^2\theta\right)^{7/2}$$

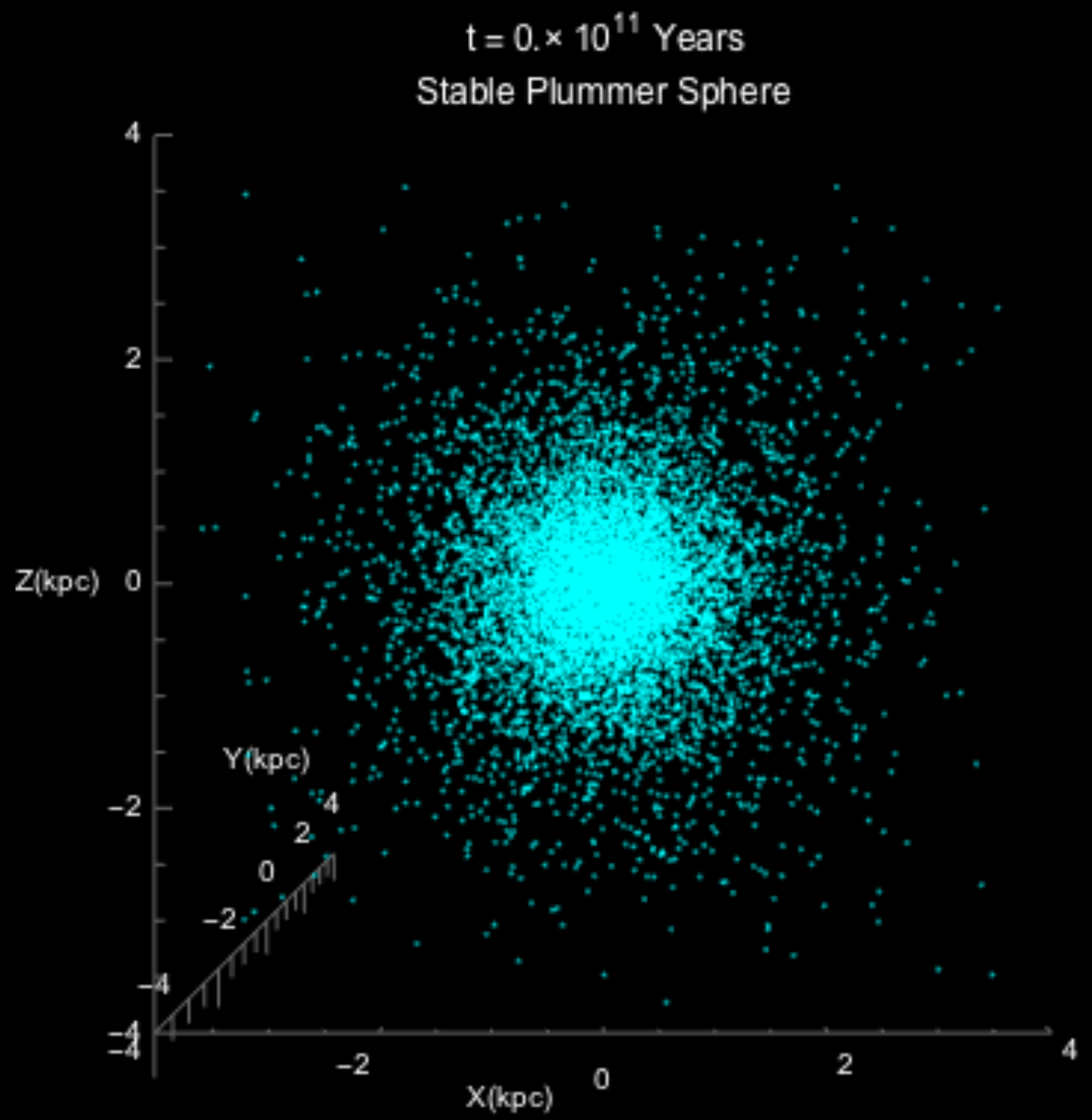
- **Quiz 2**: Solve this integral that $c_p = 2^{5/2}\pi^2 F \frac{7!!}{10!!}$ using

$$\sin^2\theta + \cos^2\theta = 1 \text{ and } \int_0^{\pi/2} \sin^{2m}\theta d\theta = \frac{\pi}{2}\frac{(2m-1)!!}{(2m)!!}$$

Steady-state solution

Credit: https://portfolium.com/entry/plummer-collapse

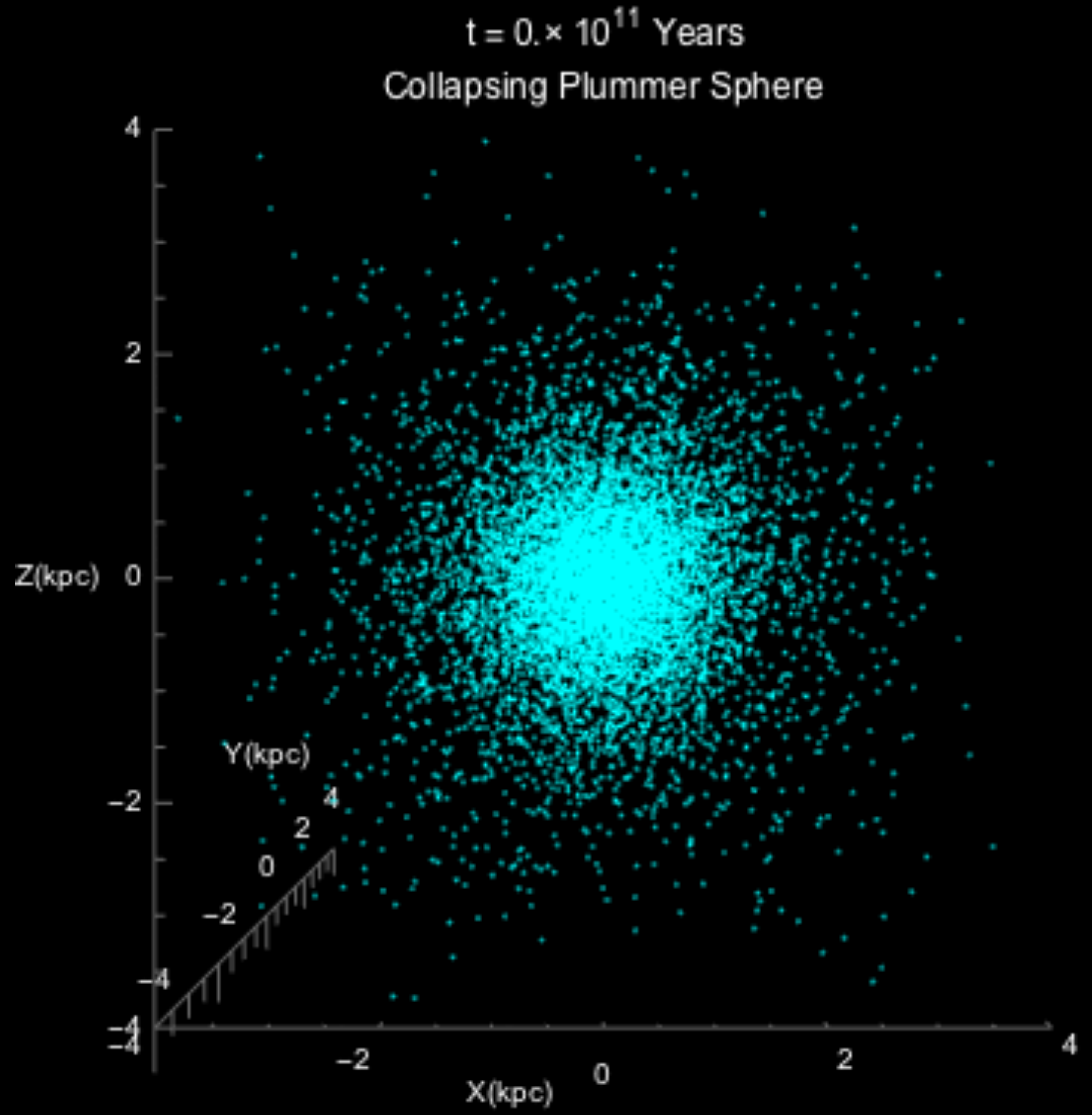
- The Plummer sphere is a steady-state solution



Collapse

Credit: https://portfolium.com/entry/plummer-collapse

- Note if change the initial conditions, you can simulate other behaviors (e.g. collapse!)



Family of solutions

Name	$\rho(r)$	$\Phi(r)$
Plummer	$\frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2}$	$\frac{-GM}{\sqrt{r^2+a^2}}$
Hernquist	$\frac{M}{2\pi}\frac{a}{r(r+a)^3}$	$\frac{-GM}{r+a}$
Jaffe	$\frac{M}{4\pi}\frac{a}{r^2(r+a)^2}$	$\frac{GM}{a}\ln\left(\frac{a}{r+a}\right)$
Gamma	$\frac{(3-\gamma)M}{4\pi a^3}\frac{a^4}{r^\gamma(r+a)^{4-\gamma}}$	$\frac{GM}{a}\begin{cases}\frac{1}{\gamma-2}\left[1-\left(\frac{r}{r+a}\right)^{2-\gamma}\right], & \gamma\neq 2\\ \ln\left(\frac{r}{r+a}\right), & \gamma=2\end{cases}$

Recap: Virial theorem

- For a system in equilibrium $2\langle K\rangle + \langle U\rangle = 0$
- Because $E = K + U$, we have $\langle K\rangle = -E, \ \ \langle U\rangle = 2E$
- For an N-body system of total mass M and total energy $E < 0$ (meaning it's gravitationally bound), we can define characteristic *virial* velocity $V_{\rm vir}$ and length scales $R_{\rm vir}$ by assuming the

$$\frac{1}{2}MV_{\rm vir}^2 = \langle K\rangle$$

$$\frac{GM^2}{R_{\rm vir}} = \langle U\rangle$$

- These give $V_{\rm vir} = \sqrt{2|E|/M}$ and $R_{\rm vir} = GM^2/(2E)$

“Hypervirial” Models

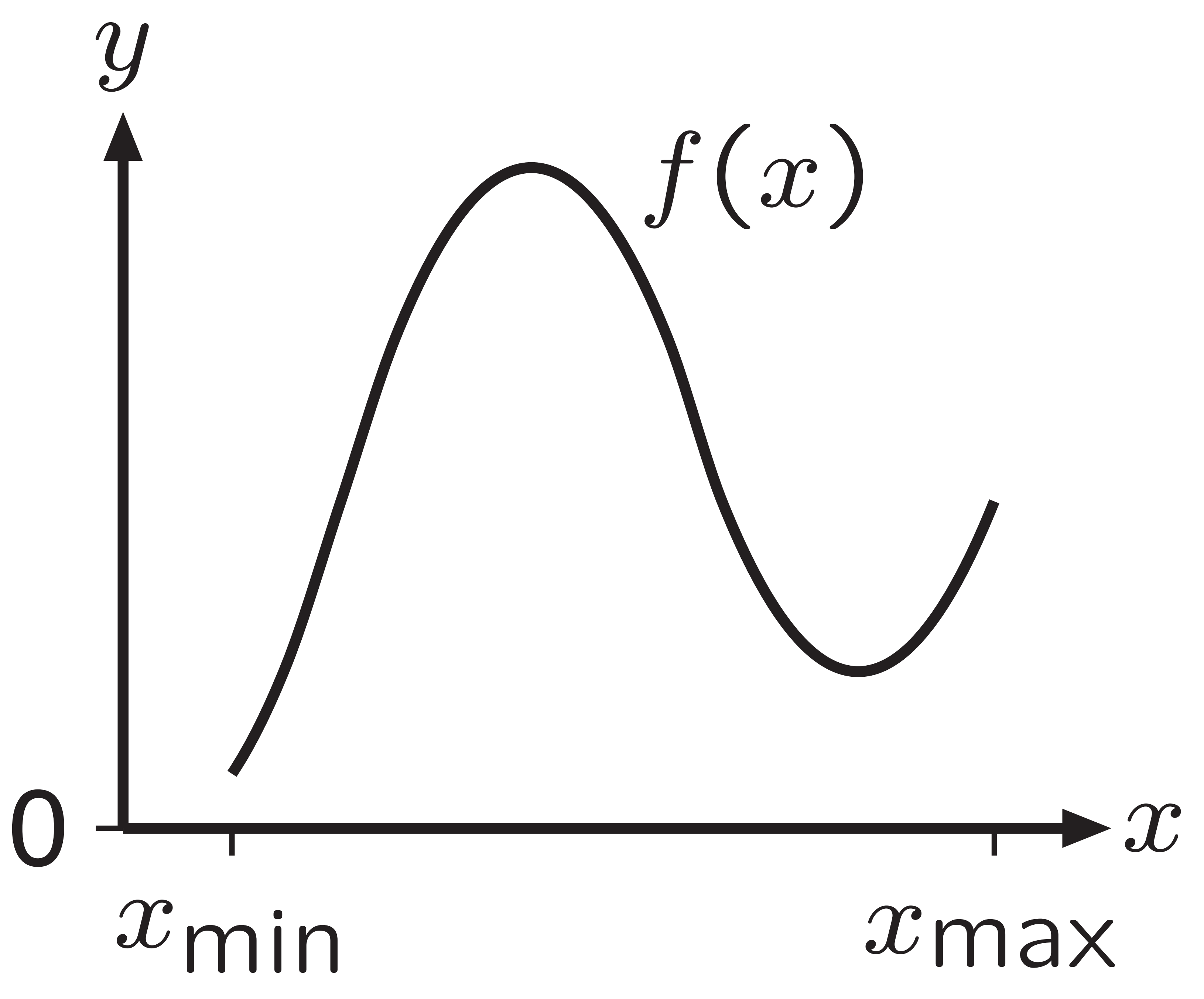
- Hypervirial models [arXiv:astro-ph/0501091] are models that obey the virial theorem at each and every point!
- Distribution functions $f \propto L^{p-2} E^{(3p+1)/2}$
- Plummer model ($p = 2$) is one example

How to initialize a Plummer sphere?

- Monte Carlo method!

Monte Carlo Methods

Assume function $f(x)$,
studied range $x_{\text{min}} < x < x_{\text{max}}$,
where $f(x) \geq 0$ everywhere
(in practice x is multidimensional)



Two standard tasks:

1) Calculate (approximatively)

$$\int_{x_{\text{min}}}^{x_{\text{max}}} f(x')\,\mathrm{d}x'$$

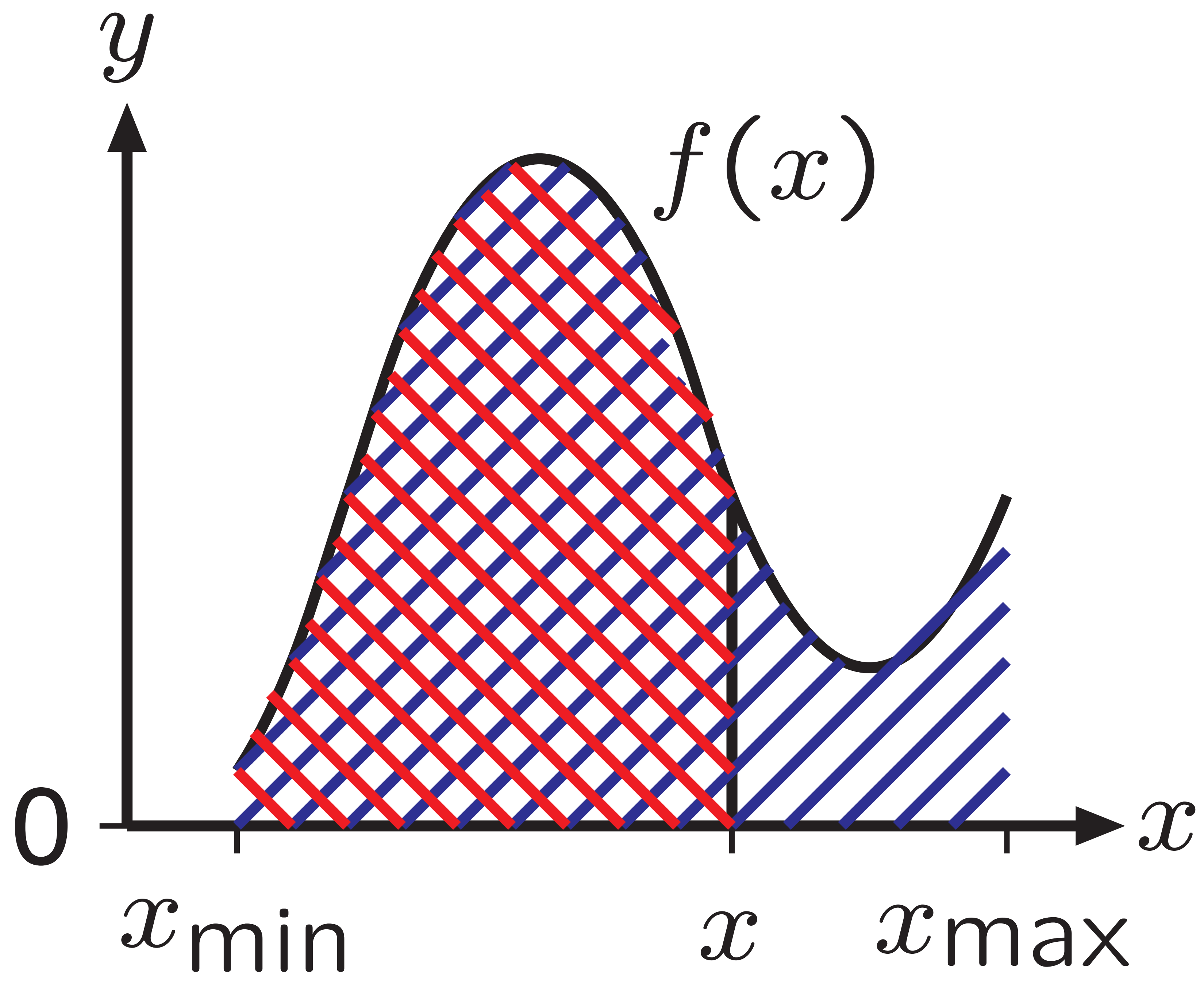
2) Select x at random according to $f(x)$

Monte Carlo Methods

Selection of x according to $f(x)$
is equivalent to uniform selection of (x, y) in the area
$x_{\text{min}} < x < x_{\text{max}}$, $0 < y < f(x)$
since $\mathcal{P}(x) \propto \int_0^{f(x)} 1 \, \mathrm{d}y = f(x)$

Therefore

$$\int_{x_{\text{min}}}^{x} f(x') \, \mathrm{d}x' = R \int_{x_{\text{min}}}^{x_{\text{max}}} f(x') \, \mathrm{d}x'$$



Monte Carlo Methods

Method 1: Analytical solution

If know primitive function $F(x)$ and know inverse $F^{-1}(y)$ then

$$\begin{aligned} F(x) - F(x_{\text{min}}) &= R\,(F(x_{\text{max}}) - F(x_{\text{min}})) = R\,A_{\text{tot}} \\ \Longrightarrow x &= F^{-1}(F(x_{\text{min}}) + R\,A_{\text{tot}}) \end{aligned}$$

Monte Carlo Methods

Method 2: Hit-and-miss

If $f(x) \leq f_{\text{max}}$ in $x_{\text{min}} < x < x_{\text{max}}$

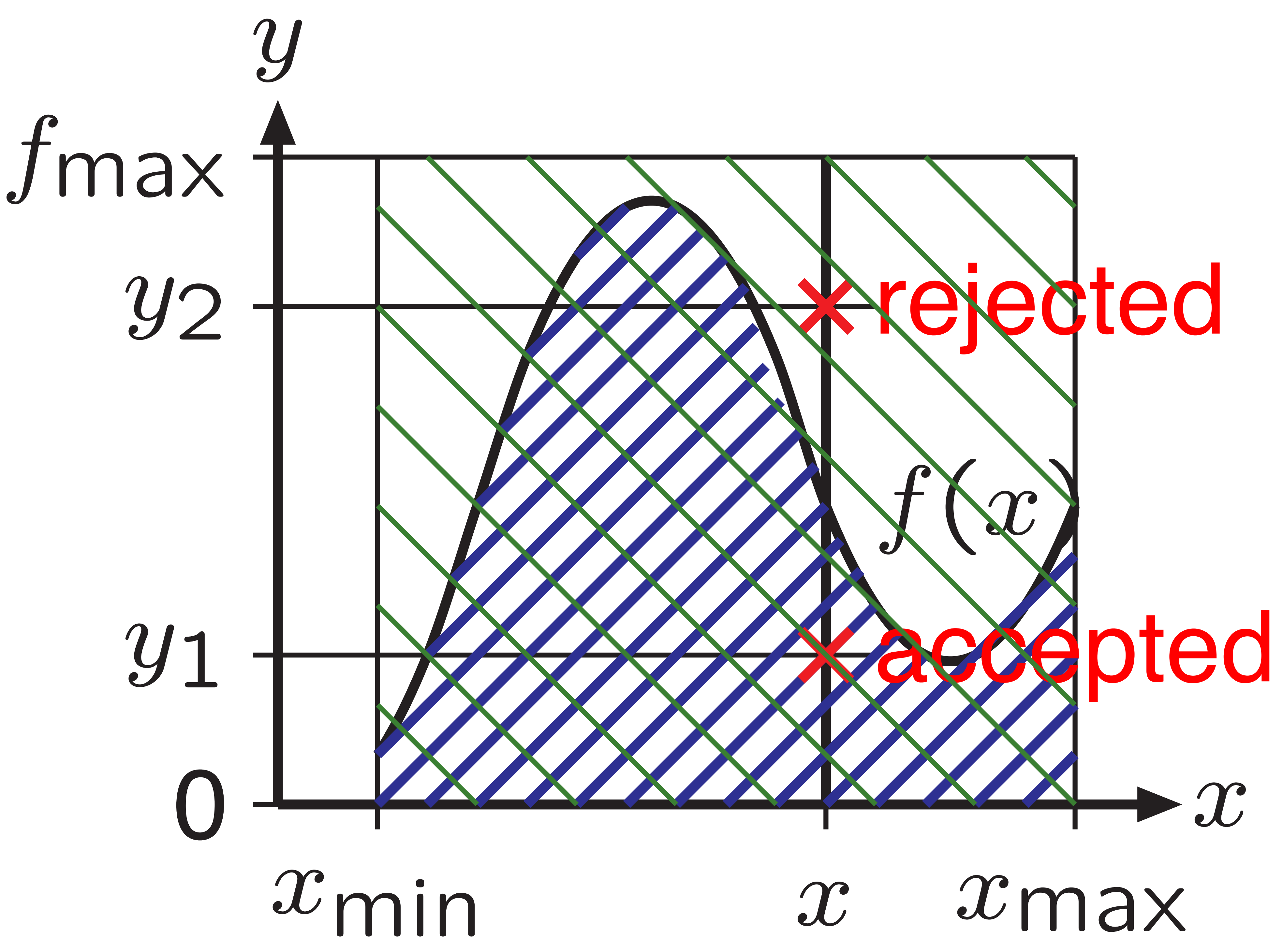
use interpretation as an area

1) select $x = x_{\text{min}} + R\,(x_{\text{max}} - x_{\text{min}})$

2) select $y = R\,f_{\text{max}}$ (new R!)

3) while $y > f(x)$ cycle to 1)

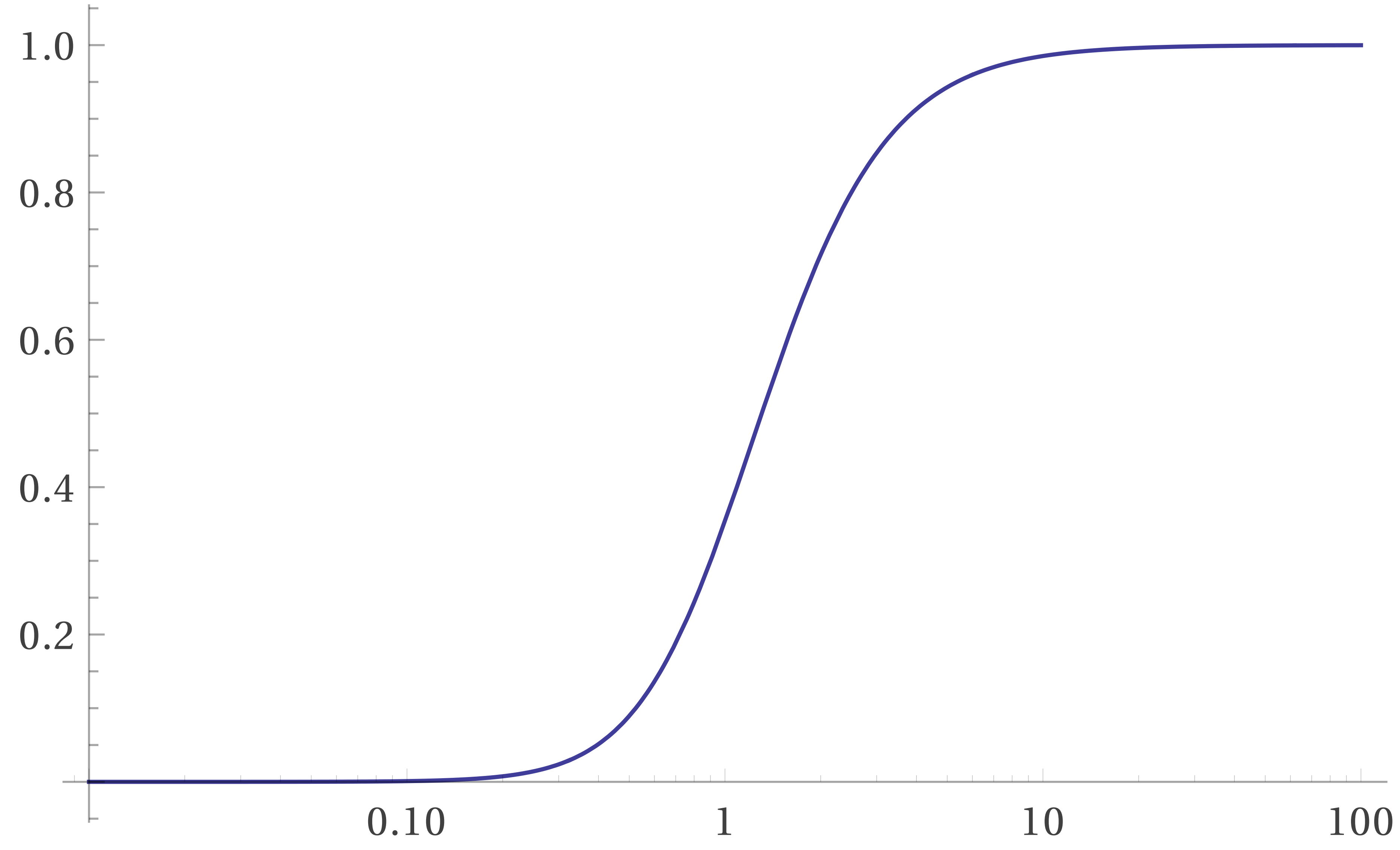
Integral as by-product:



Generate a Plummer sphere with MC

- Take units where $G = 1$, $a = 1$, and $M = 1$ for convenience
- Consider N equal mass stars $m = 1/N$
- We can find out what the mass $M(r)$ within a sphere of radius r is in the Plummer model

$$M(r) = \int_0^r 4\pi r'^2 dr' \rho(r') = \frac{r^3}{(r^2 + 1^2)^{3/2}}$$

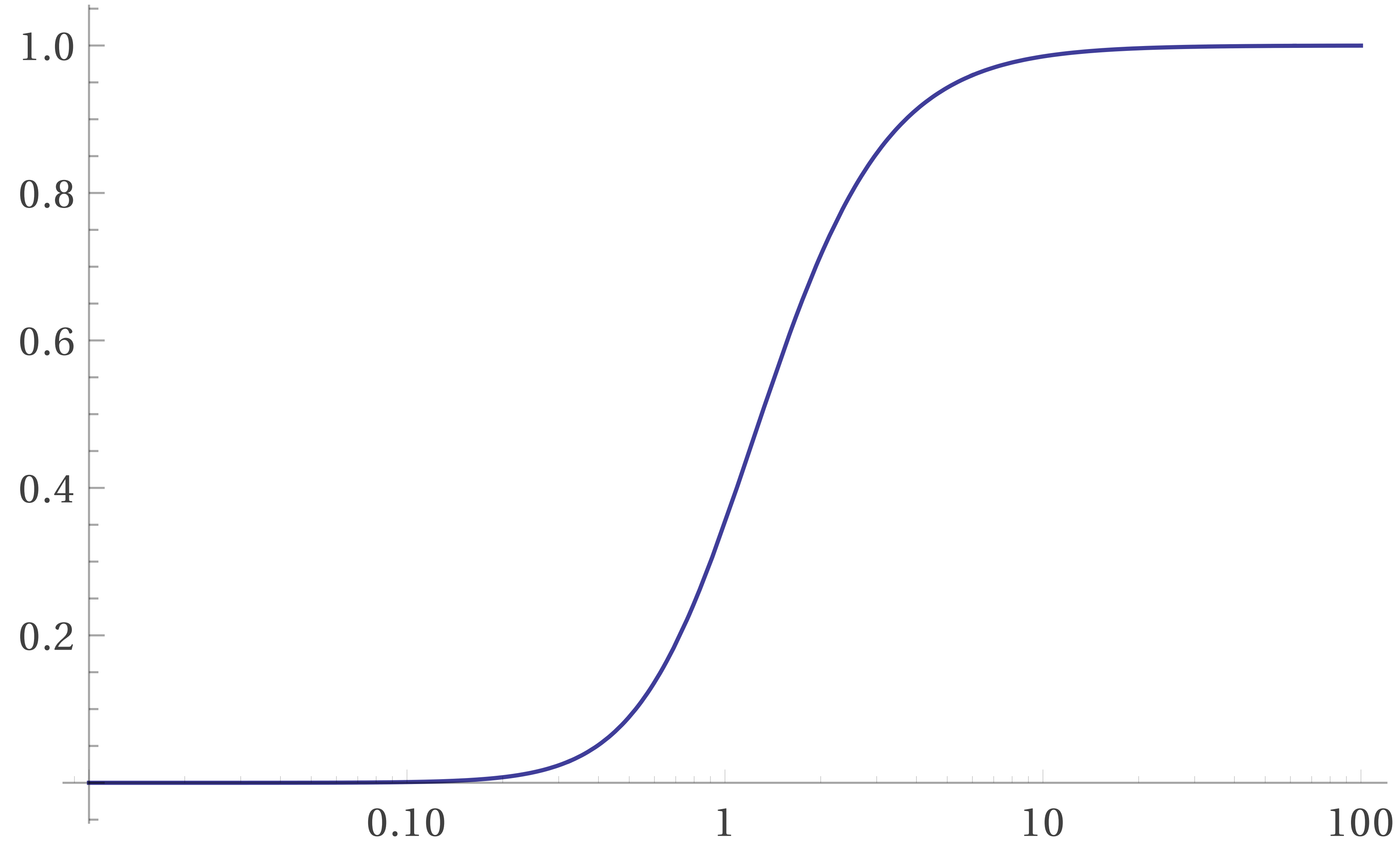


- How do we generate points according to this distribution?

Generate a Plummer sphere with MC

- Assume we have a way to get random numbers $X \in [0,1]$
- Simply generate X_1 then equate $M(r) = X_1$ to solve

$$r = (X_1^{-2/3} - 1)^{-1/2}$$

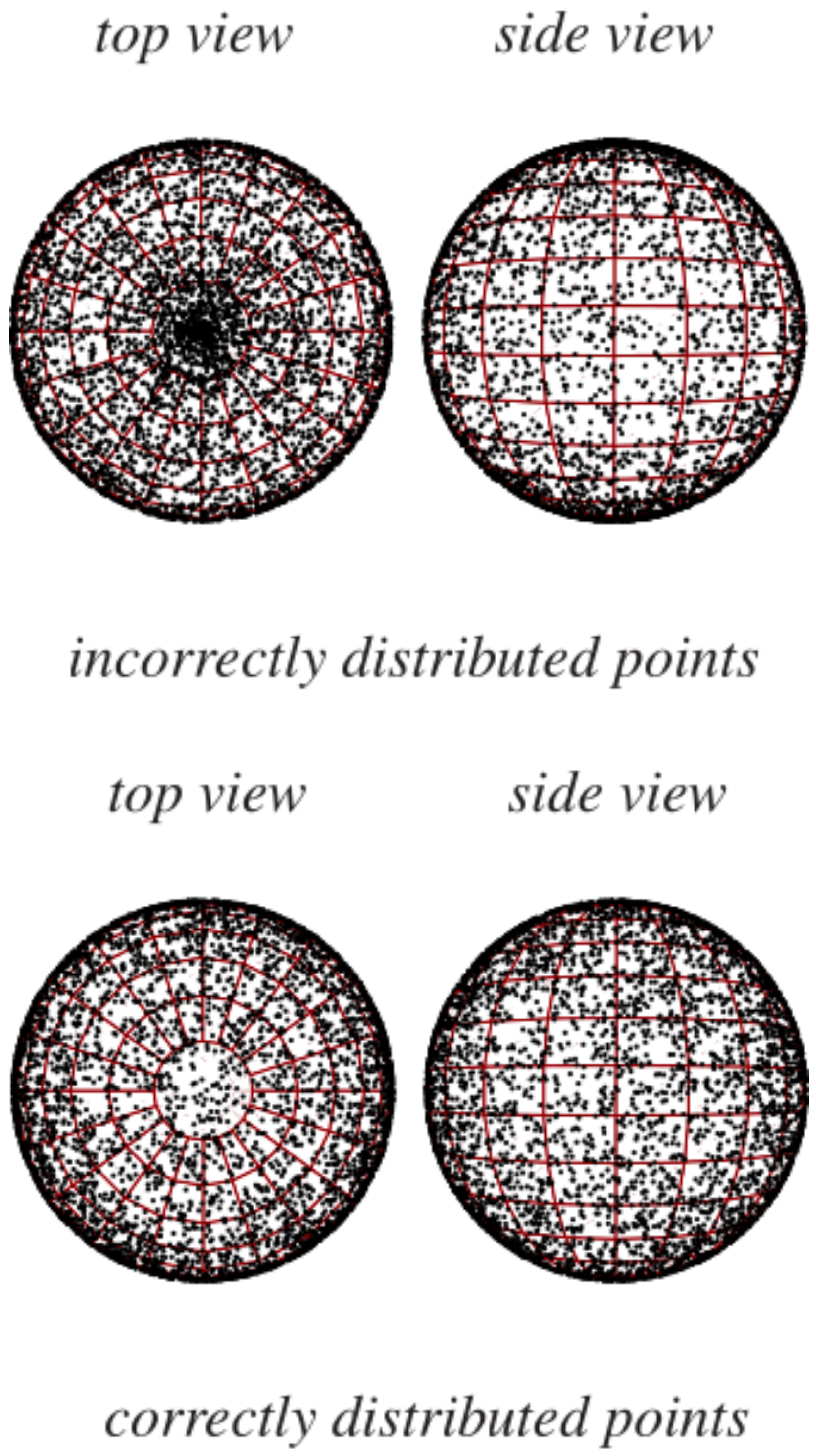


Generate a Plummer sphere with MC

- Next, we need to find the actual position (x, y, z) should be selected on the sphere of radius r
    - Careful! Can't just uniformly sample $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ because $d\Omega = \sin\theta d\phi d\theta = d\phi d(\cos\theta)$ but we ***can*** uniformly sample $\cos\theta \in [-1,1]$
    - See: https://mathworld.wolfram.com/SpherePointPicking.html
- Two random numbers $X_2 = \cos\theta$ and $X_3 = \phi/(2\pi)$ can be interpreted as angles

$$z = (1 - 2X_2)r, \quad x = (r^2 - z^2)^{1/2}\cos(2\pi X_3)$$

$$y = (r^2 - z^2)^{1/2}\sin(2\pi X_3)$$



Generate velocities

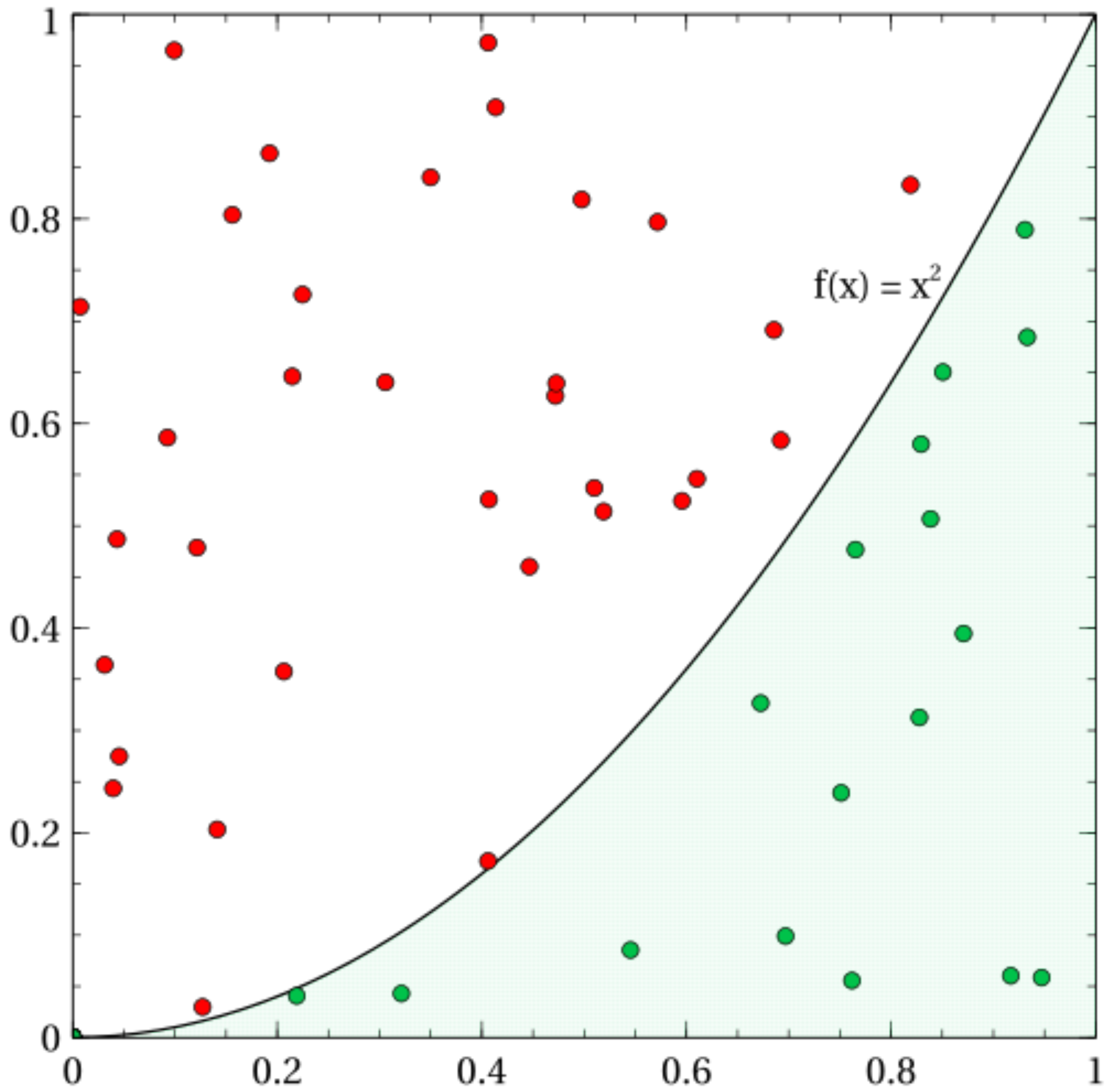
- Maximum value of v at distance r is the escape velocity

$$v_e = \sqrt{-2\Phi} = 2^{1/2}(1 + r^2)^{-1/4}$$

- Writing $q = v/v_e$, the probability distribution is given by

$$\int d^3 r f(r, v) \propto g(q) = q^2(1 - q^2)^{7/2}$$

- Now this is hard to analytically integrate, so let's use hit-and-miss! Note $q \in [0,1]$ and $g(q) \in [0,0.1]$
- Two random numbers X_4 and X_5
  - If $0.1X_5 \ < g(X_4)$, hit! Keep $q = X_4$
  - Else, miss! Generate again



We consider a convenient "model problem" defined as follows. At time $t=0$ the space density $\varrho(\boldsymbol{r}, t)$ of the cluster conforms to Plummer's model, i.e. a polytrope of index 5:

$$\varrho(\boldsymbol{r}, 0) = (3/4\pi)\, M R^{-3}\, [1 + (r/R)^2]^{-5/2}\,, \tag{1}$$

where M is the total mass of the cluster, and R is a parameter which determines the dimensions of the cluster. The gravitational potential is then

$$U(\boldsymbol{r}, 0) = -GMR^{-1}\, [1 + (r/R)^2]^{-1/2} \tag{2}$$

and the potential energy of the cluster is

$$W = -(3\pi/32)\, GM^2 R^{-1}\,. \tag{3}$$

Initially the system is assumed to be in a steady state, with a velocity distribution everywhere isotropic. This implies that the initial distribution function is given by

$$f(\boldsymbol{r}, \boldsymbol{V}, 0) = \begin{cases} (24\sqrt{2}/7\pi^3)\, G^{-5} M^{-4} R^2 (-E)^{7/2} & \text{for} \quad E < 0\,, \\ 0 & \text{for} \quad E > 0\,. \end{cases} \tag{4}$$

Here $f(\boldsymbol{r}, \boldsymbol{V}, t)\, d\boldsymbol{r}\, d\boldsymbol{V}$ is the total mass of the stars with position $\boldsymbol{r}$ and velocity $\boldsymbol{V}$, at time t, and E is the energy per unit mass of a star:

$$E = U + V^2/2\,. \tag{5}$$

The total energy of the system is then

$$\mathscr{E} = W/2 = -(3\pi/64)\, GM^2 R^{-1}\,. \tag{6}$$

Lecture and Lab note on Plummer's model

Instructions to construct the phase space distribution are given in the Appendix

Virial Theorem?

Appendix: Generation of Initial Coordinates

Plummer's model was found to be convenient for a comparison of methods, and might be adopted as a standard model for such comparisons. Therefore we think it useful to give here a detailed prescription for the construction of initial positions and velocities.

There must be available a subroutine which generates normalized random numbers X, with uniform probability distribution between 0 and 1. We consider the system as defined by (1), (2) and (4), taking $G=1$, $M=1$, $R=1$ for convenience. In the equal-mass case, each star then has a mass $m=1/N$. From (1), the mass inside a sphere of radius r is

$$M(r)=r^3(1+r^2)^{-3/2}. \tag{A1}$$

In order to select a value of r for a star, we simply generate a random number X_1 and equate $M(r)$ to X_1, so that r is given by

$$r=(X_1^{-2/3}-1)^{-1/2}. \tag{A2}$$

The actual position (x, y, z) of the star should now be selected on the sphere of radius r, with uniform probability. This is done by the usual trick: we generate two normalized random numbers X_2 and X_3 and compute

$$\begin{aligned} &z=(1-2X_2)\,r, \quad x=(r^2-z^2)^{1/2}\cos 2\pi X_3, \\ &y=(r^2-z^2)^{1/2}\sin 2\pi X_3. \end{aligned} \tag{A3}$$

Next, we compute the velocity modulus for the same star. The maximum value of V at distance r from the centre is the escape velocity

$$V_e=(-2U)^{1/2}=2^{1/2}(1+r^2)^{-1/4}. \tag{A4}$$

We write $V/V_e=q$. Then (4) shows that the probability distribution of q is proportional to

$$g(q)=q^2(1-q^2)^{7/2}. \tag{A5}$$

A convenient way to sample q according to this distribution is provided by von Neumann's rejection technique. Possible values of q range from 0 to 1, and $g(q)$ is always less than 0.1. Therefore we generate two normalized random numbers X_4 and X_5; if $0.1\,X_5<g(X_4)$, we adopt $q=X_4$; if not, a new pair of random numbers is tried, until one is found which satisfies the inequality. The velocity modulus is then obtained, using (A4). Since the velocity distribution is isotropic, the three velocity coordinates u, v, w are computed from V in the same way as the three space coordinates from r, using two new random numbers X_6 and X_7:

$$\begin{aligned} &w=(1-2X_6)\,V, \quad u=(V^2-w^2)^{1/2}\cos 2\pi X_7, \\ &v=(V^2-w^2)^{1/2}\sin 2\pi X_7. \end{aligned} \tag{A6}$$

The whole procedure is repeated for each of the N stars. Finally, the values of m, x, y, z, u, v, w may be scaled to suit the numerical scheme used. If a cluster with mass M and energy $\mathscr{E}$ is desired, while keeping $G=1$, then masses should be multiplied by M, lengths by $(3\pi/64)\,M^2|\mathscr{E}|^{-1}$, and velocities by $(64/3\pi)\,|\mathscr{E}|^{1/2}\,M^{-1/2}$.