PHYS 141/241

Lecture 10: Plummer Model and Monte Carlo (Continued)

Javier Duarte — April 21, 2023

Recap: Plummer Model

- Distribution function

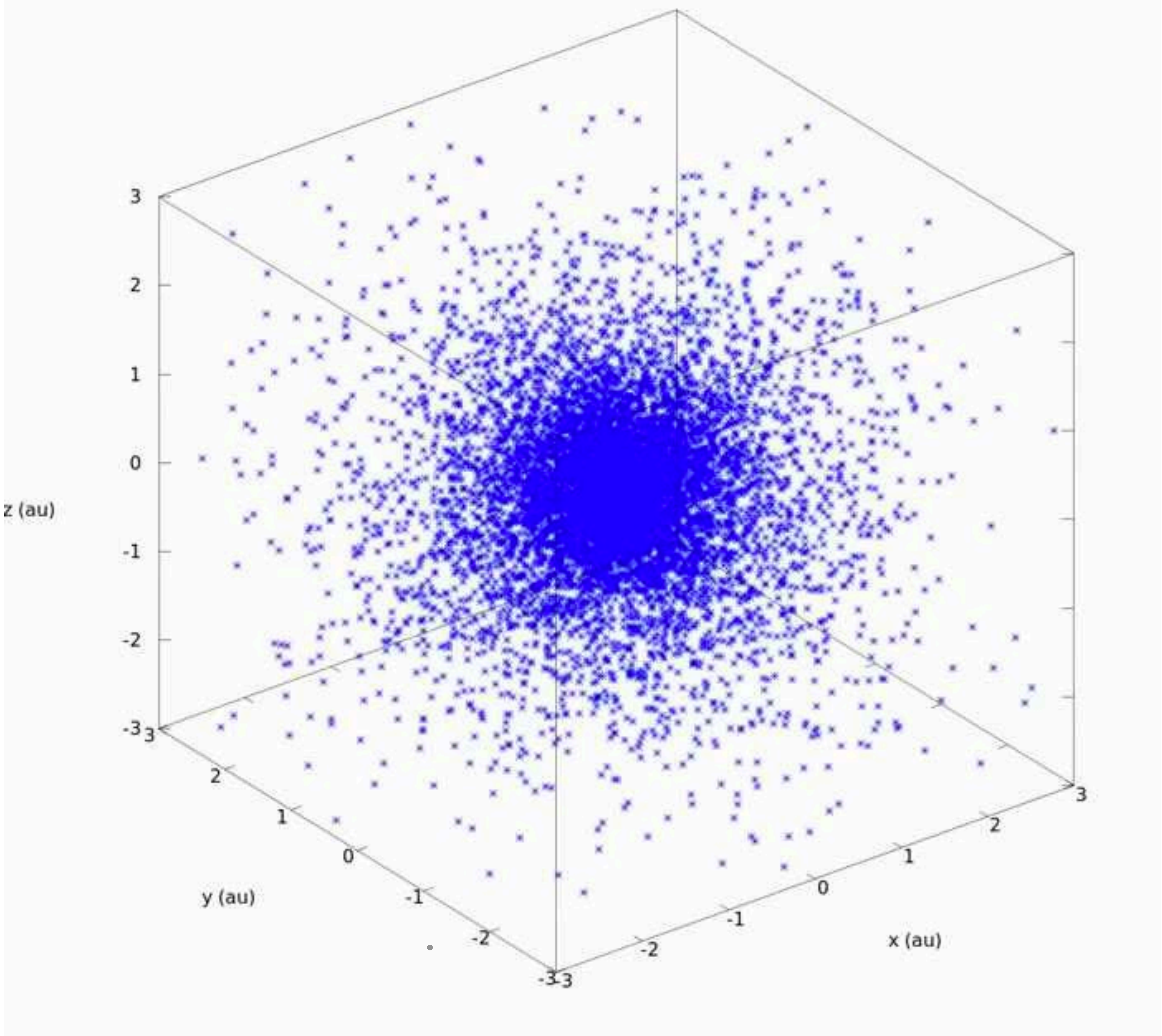
$$f = \begin{cases} F(-E)^{7/2} & \text{if } E < 0 \\ 0 & \text{if } E \geq 0 \end{cases}$$

where $E = \frac{1}{2}v^2 + \Phi(r)$

- We found

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1 + \frac{r^2}{a^2}\right)^{-5/2}$$

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + a^2}}$$

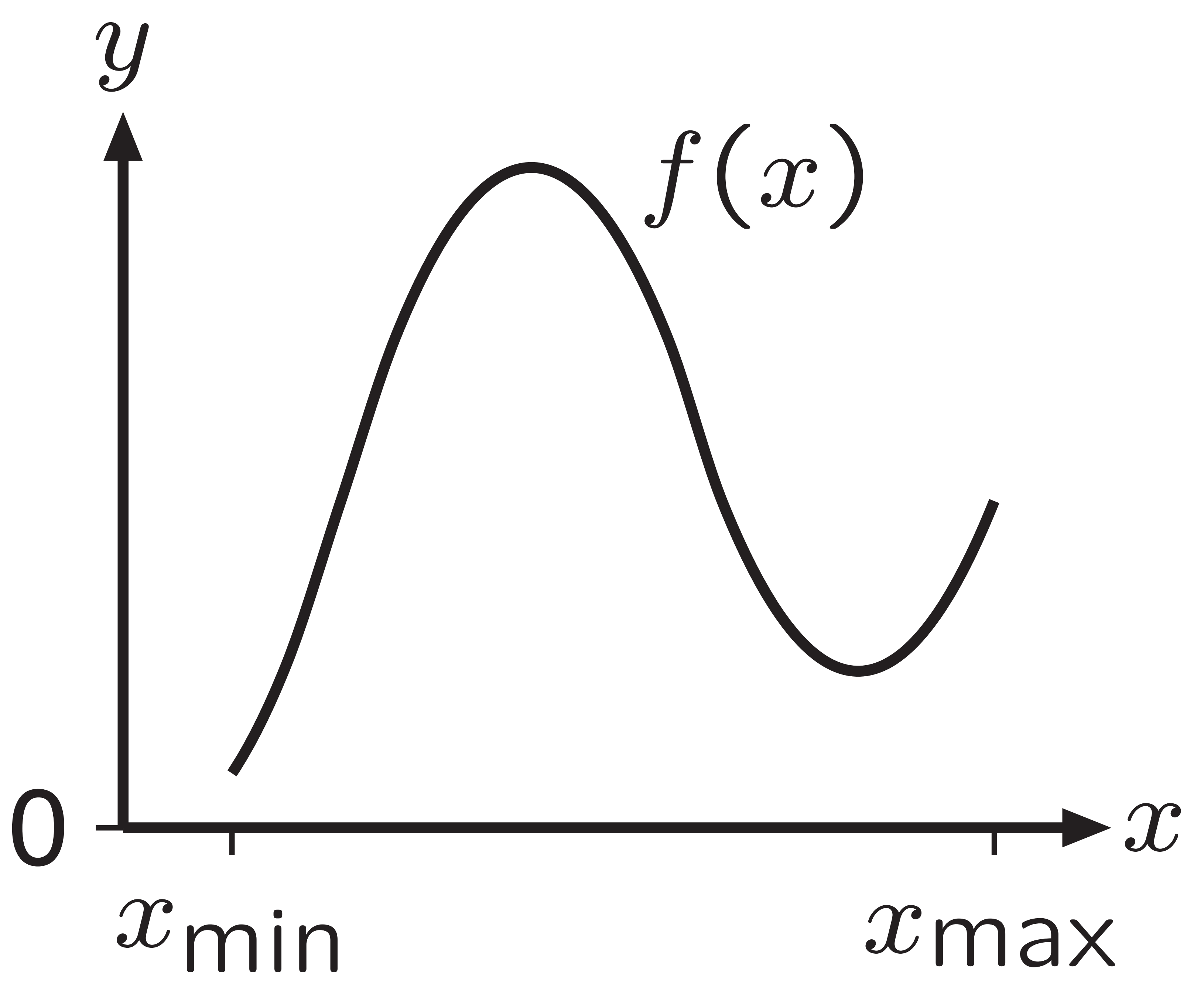


How to initialize a Plummer sphere?

- Monte Carlo method!

Monte Carlo Methods

Assume function $f(x)$,
studied range $x_{\text{min}} < x < x_{\text{max}}$,
where $f(x) \geq 0$ everywhere
(in practice x is multidimensional)



Two standard tasks:

1) Calculate (approximatively)

$$\int_{x_{\text{min}}}^{x_{\text{max}}} f(x')\,\mathrm{d}x'$$

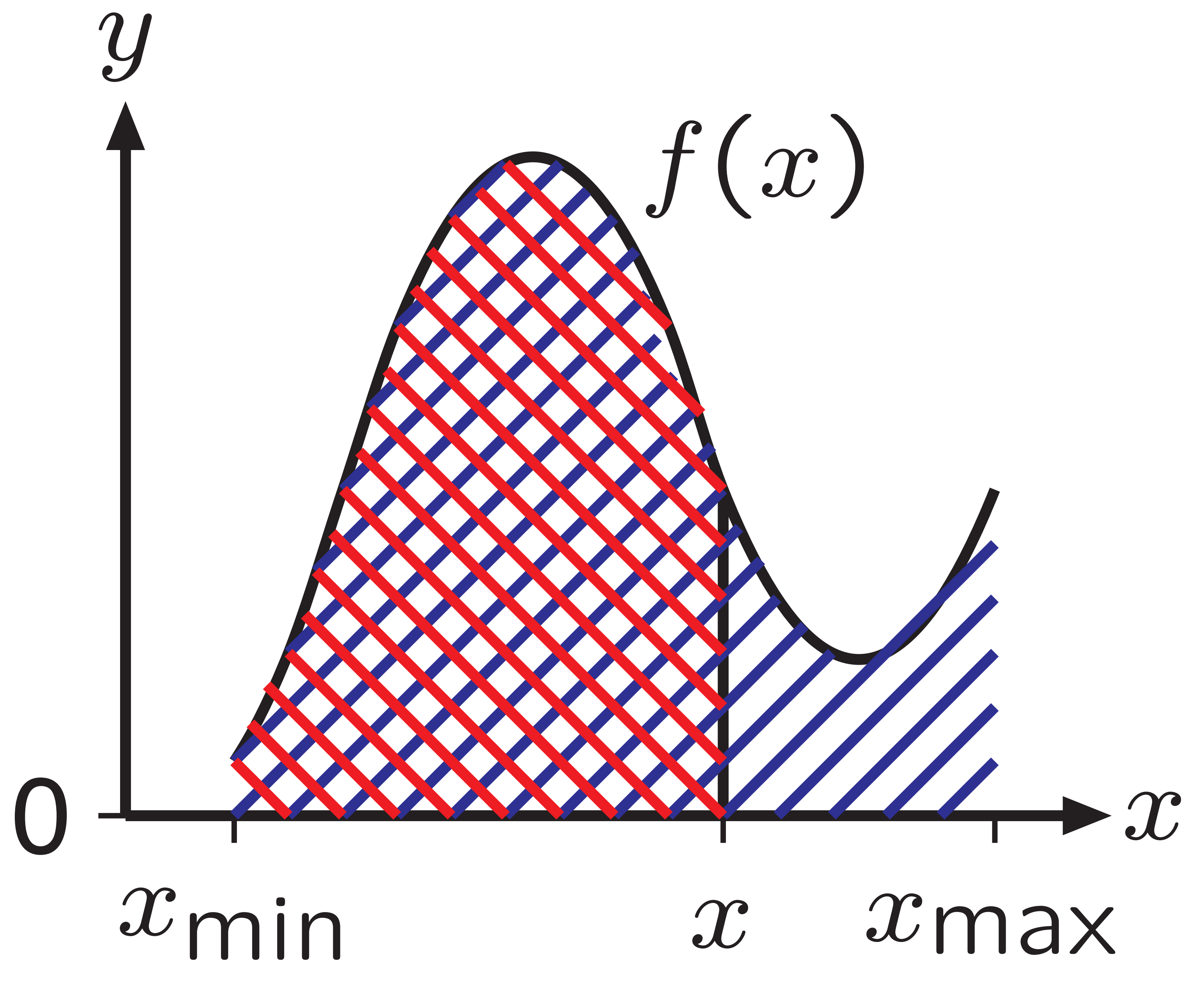
2) Select x at random according to $f(x)$

Monte Carlo Methods

Selection of x according to $f(x)$
is equivalent to uniform selection of (x, y) in the area
$x_{\mathsf{min}} < x < x_{\mathsf{max}}$, $0 < y < f(x)$
since $\mathcal{P}(x) \propto \int_0^{f(x)} 1 \,\mathrm{d}y = f(x)$

Therefore

$$\int_{x_{\mathsf{min}}}^{x} f(x')\,\mathrm{d}x' = R \int_{x_{\mathsf{min}}}^{x_{\mathsf{max}}} f(x')\,\mathrm{d}x'$$



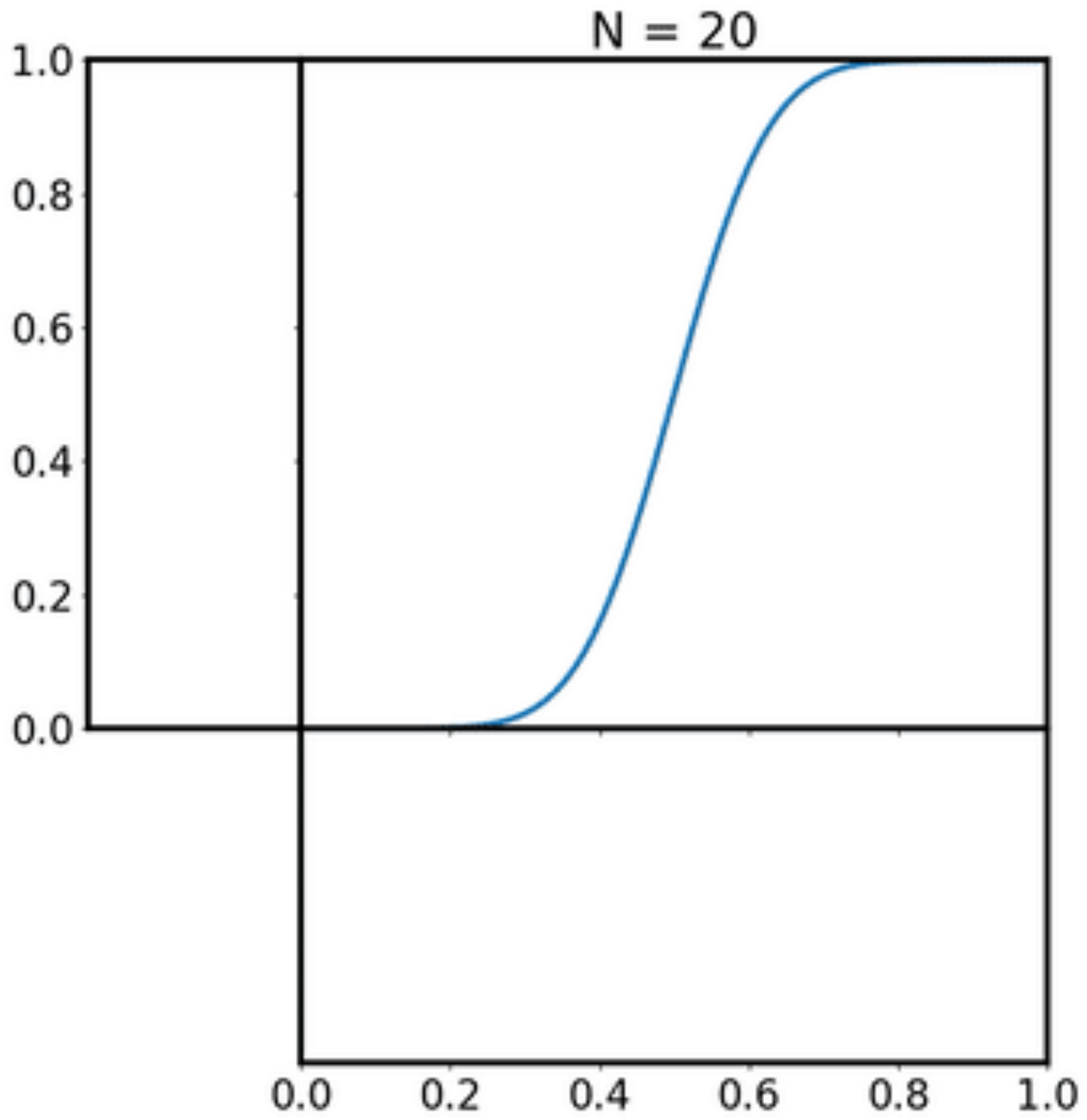
Monte Carlo Methods

Method 1: Analytical solution

If know primitive function $F(x)$ and know inverse $F^{-1}(y)$ then

$$F(x) - F(x_{\text{min}}) \;=\; R\,(F(x_{\text{max}}) - F(x_{\text{min}})) = R\,A_{\text{tot}}$$
$$\Longrightarrow x \;=\; F^{-1}(F(x_{\text{min}}) + R\,A_{\text{tot}})$$

"Inverse transform sampling"
Example: Gaussian distribution



Monte Carlo Methods

Method 2: Hit-and-miss

If $f(x) \leq f_{\text{max}}$ in $x_{\text{min}} < x < x_{\text{max}}$

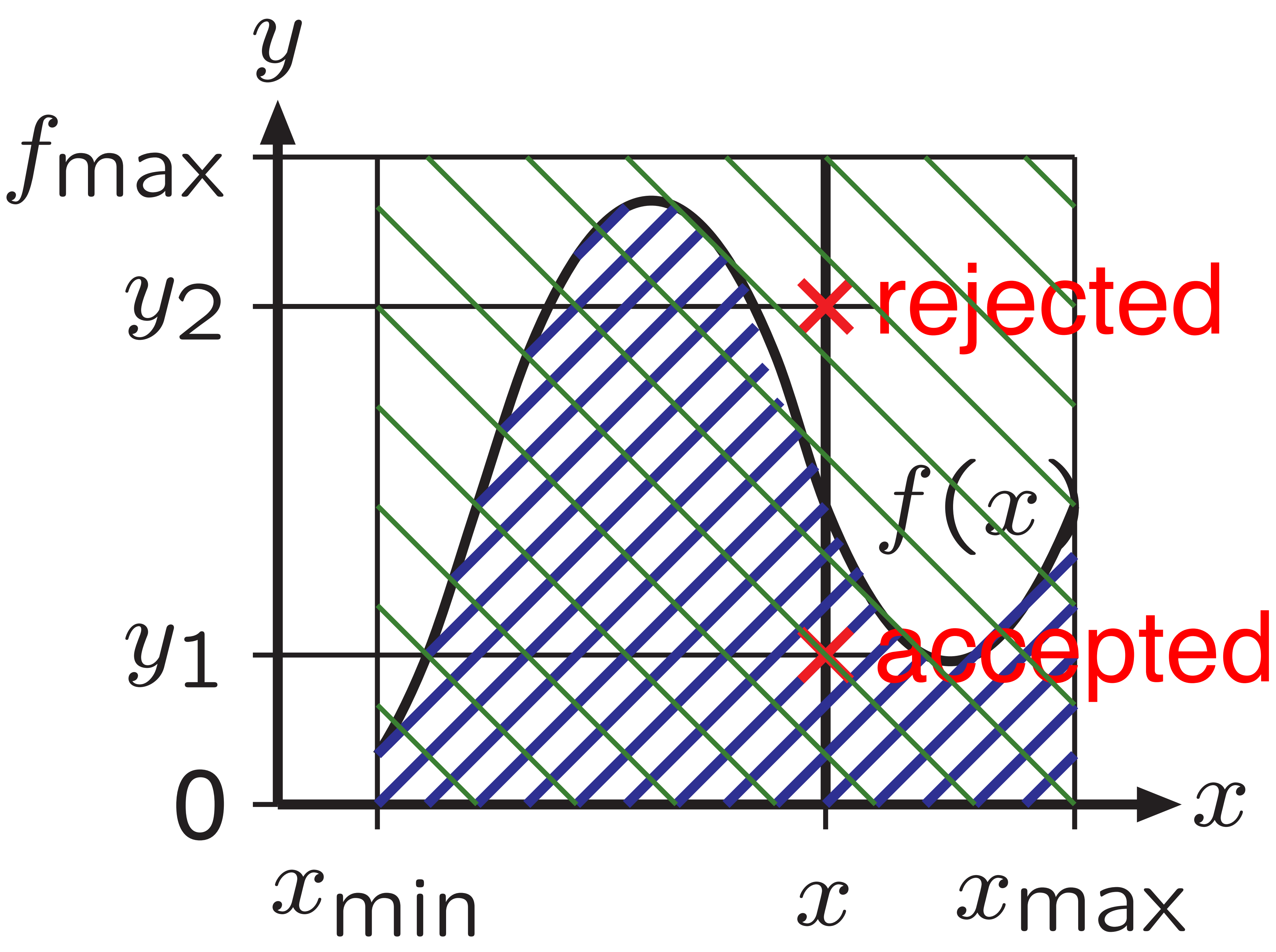
use interpretation as an area

1) select $x = x_{\text{min}} + R\,(x_{\text{max}} - x_{\text{min}})$

2) select $y = R\, f_{\text{max}}$ (new R!)

3) while $y > f(x)$ cycle to 1)

Integral as by-product:

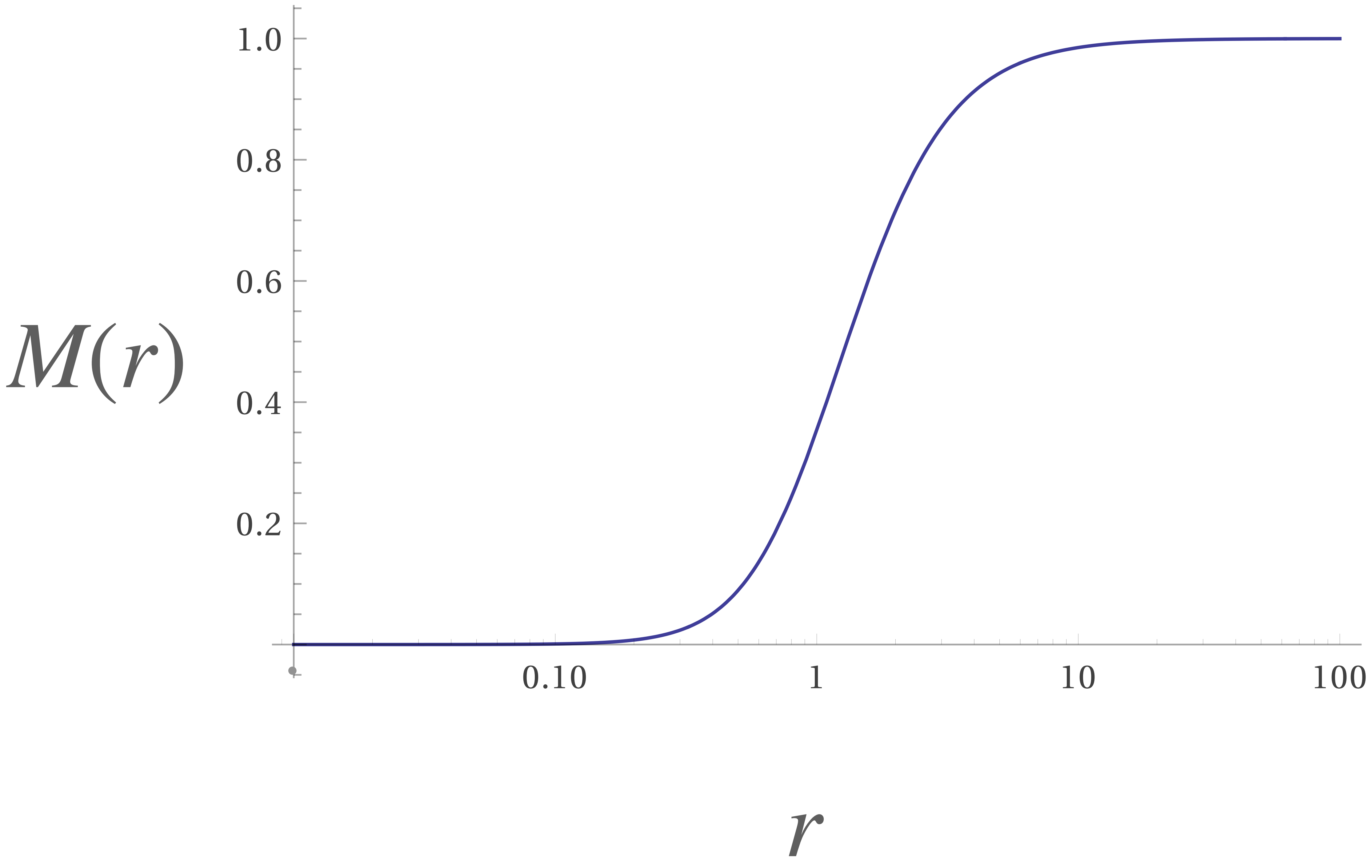


Generate a Plummer sphere with MC

- Take units where $G = 1$, $a = 1$, and $M = 1$ for convenience
- Consider N equal mass stars $m = 1/N$
- We can find out what the mass $M(r)$ within a sphere of radius r is in the Plummer model

$$M(r) = \int_0^r 4\pi r'^2 dr' \rho(r') = \frac{r^3}{(r^2+1)^{3/2}}$$

- How do we generate points according to this distribution?



Generate a Plummer sphere with MC

- Assume we have a way to get random numbers $X \in [0,1]$
- Simply generate X_1 then equate $M(r) = X_1$ then solve for r

$$M(r) = X_1 = \frac{r^3}{(r^2+1)^{3/2}}$$

$$= \left(1 + r^{-2}\right)^{-3/2}$$

- Thus,

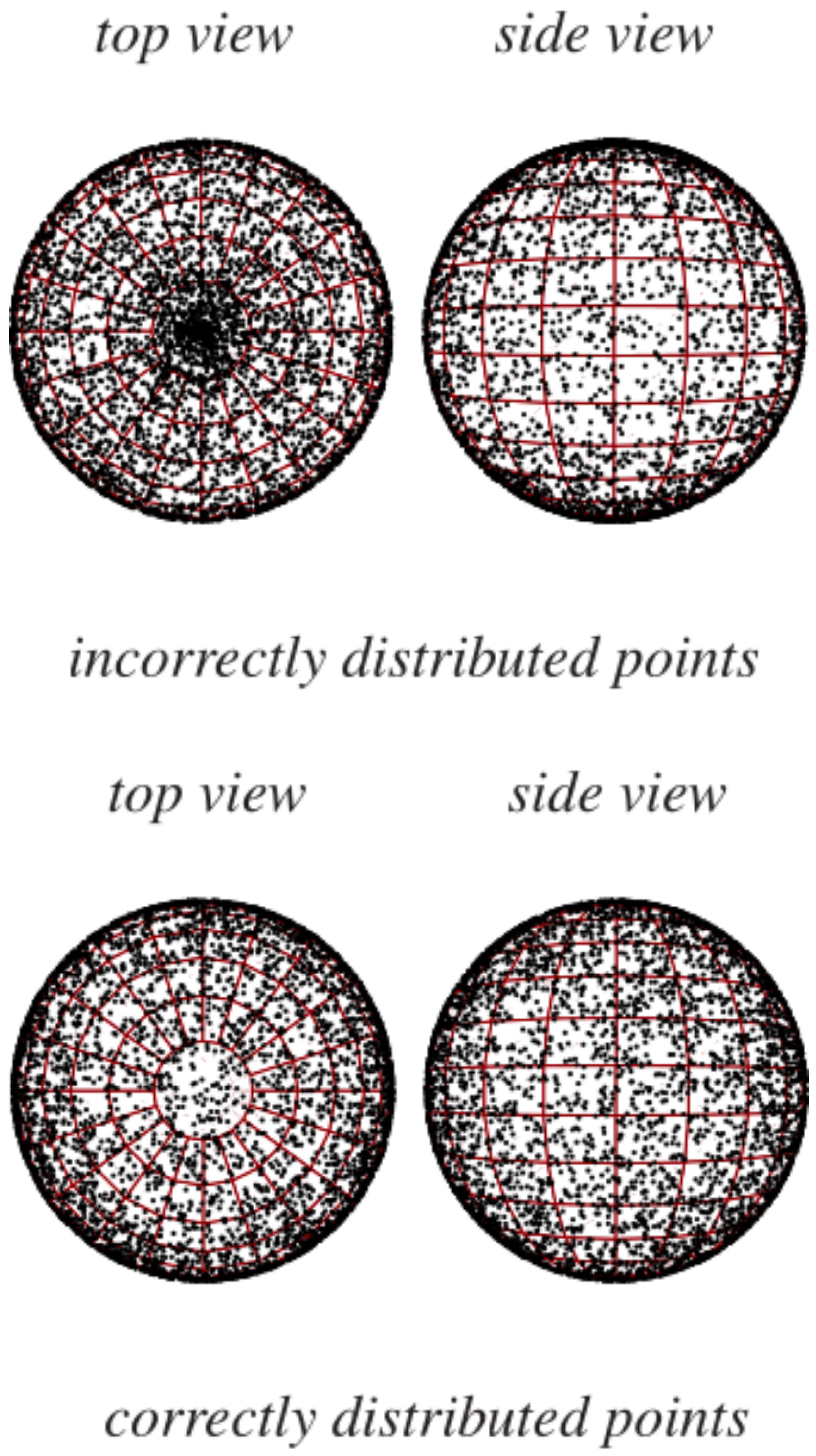
$$r = (X_1^{-2/3} - 1)^{-1/2}$$

Generate a Plummer sphere with MC

- Next, we need to find the actual position (x, y, z) should be selected on the sphere of radius r
  - Careful! Can't just uniformly sample $\theta \in [0,\pi]$ and $\phi \in [0,2\pi]$ because $d\Omega = \sin\theta d\phi d\theta = d\phi d(\cos\theta)$ but we ***can*** uniformly sample $\cos\theta \in [-1,1]$
  - See: https://mathworld.wolfram.com/SpherePointPicking.html
- Two random numbers $X_2 = \cos\theta$ and $X_3 = \phi/(2\pi)$

$$z = (1 - 2X_2)r, \quad x = (r^2 - z^2)^{1/2}\cos(2\pi X_3)$$

$$y = (r^2 - z^2)^{1/2}\sin(2\pi X_3)$$



Generate a Plummer sphere with MC

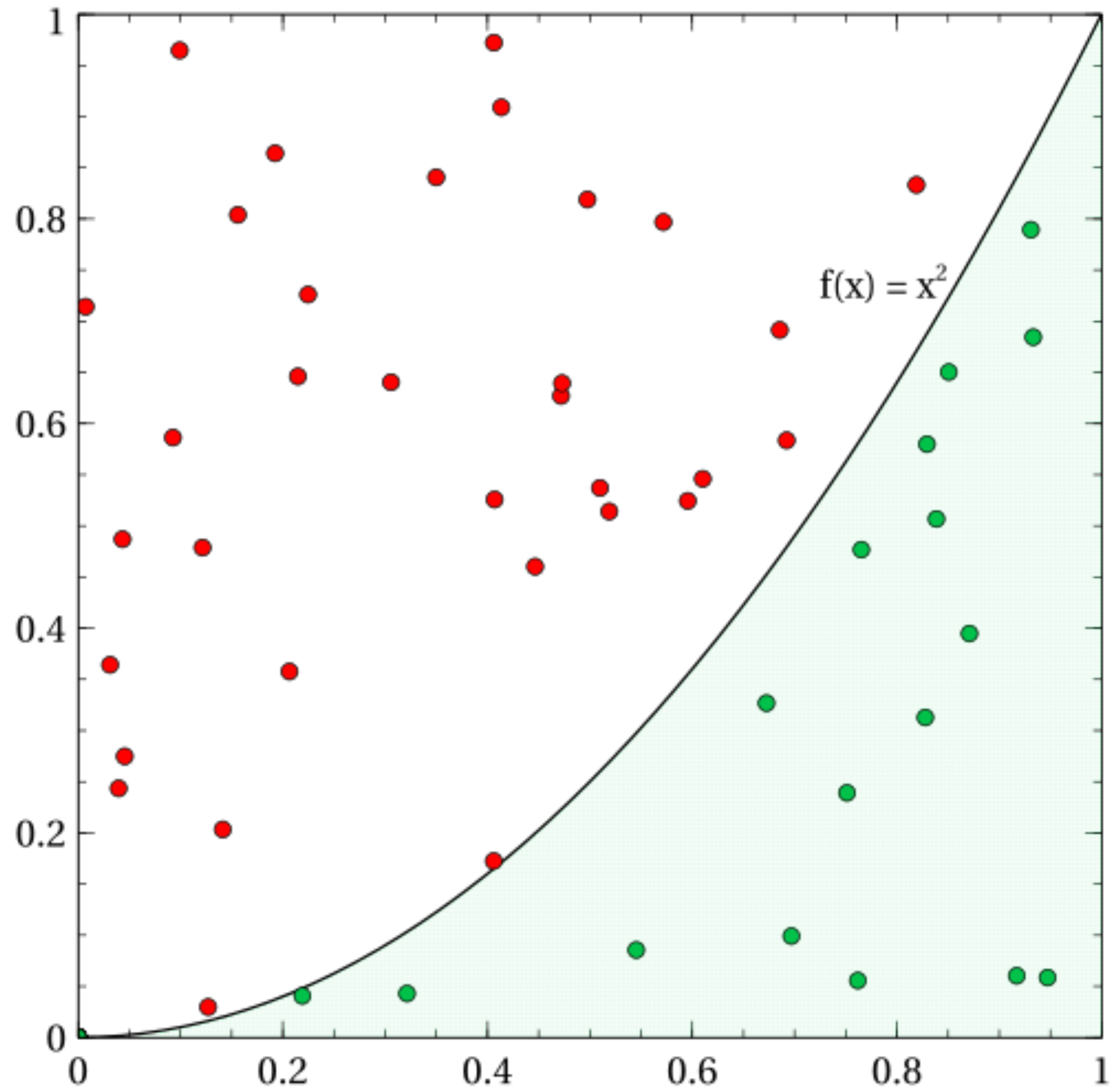
- Maximum value of v at distance r is the escape velocity

$$v_e = \sqrt{-2\Phi} = 2^{1/2}(1 + r^2)^{-1/4}$$

- Writing $q = v/v_e$, the probability distribution is given by

$$\int d^3 r f(r, v) \propto g(q) = q^2(1 - q^2)^{7/2}$$

- Now this is hard to analytically integrate, so let's use hit-and-miss! Note $q \in [0,1]$ and $g(q) \in [0,0.1]$
- Two random numbers X_4 and X_5
    - If $0.1X_5 \;\; < g(X_4)$, hit! Keep $q = X_4$
    - Else, miss! Generate again



Generate a Plummer sphere with MC

- Velocities are isotropic, so we can use the same trick
- Two random numbers again X_6 and X_7

$$v_z = (1 - 2X_6)v$$

$$v_x = (v^2 - v_z^2)^{1/2}\cos(2\pi X_7)$$

$$v_y = (v^2 - v_z^2)^{1/2}\sin(2\pi X_7)$$

Generate a Plummer sphere with MC

- Restoring units: if a cluster with mass M and energy total energy $\mathscr{E}$ is desired with $G = 1$, then
  - masses can be scaled by M
  - lengths can be scaled by $(3\pi/64)M^2 |\mathscr{E}|^{-1}$
  - velocities can be scaled by $(64/(3\pi))|\mathscr{E}|^{1/2}$

Alternative method

- Alternatively, we can just use hit-and-miss method directly on the full distribution function!

$$f(r, v) = \begin{cases} F(-E)^{7/2} & \text{if } E < 0 \\ 0 & \text{if } E \geq 0 \end{cases}$$

We consider a convenient "model problem" defined as follows. At time $t=0$ the space density $\varrho(\boldsymbol{r}, t)$ of the cluster conforms to Plummer's model, i.e. a polytrope of index 5:

$$\varrho(\boldsymbol{r}, 0) = (3/4\pi)\, M R^{-3}\, [1 + (r/R)^2]^{-5/2}\,, \tag{1}$$

where M is the total mass of the cluster, and R is a parameter which determines the dimensions of the cluster. The gravitational potential is then

$$U(\boldsymbol{r}, 0) = -G M R^{-1}\, [1 + (r/R)^2]^{-1/2} \tag{2}$$

and the potential energy of the cluster is

$$W = -(3\pi/32)\, G M^2 R^{-1}\,. \tag{3}$$

Initially the system is assumed to be in a steady state, with a velocity distribution everywhere isotropic. This implies that the initial distribution function is given by

$$f(\boldsymbol{r}, \boldsymbol{V}, 0) = \begin{cases} (24\sqrt{2}/7\pi^3)\, G^{-5} M^{-4} R^2 (-E)^{7/2} & \text{for} \quad E < 0\,, \\ 0 & \text{for} \quad E > 0\,. \end{cases} \tag{4}$$

Here $f(\boldsymbol{r}, \boldsymbol{V}, t)\, d\boldsymbol{r}\, d\boldsymbol{V}$ is the total mass of the stars with position $\boldsymbol{r}$ and velocity $\boldsymbol{V}$, at time t, and E is the energy per unit mass of a star:

$$E = U + V^2/2\,. \tag{5}$$

The total energy of the system is then

$$\mathscr{E} = W/2 = -(3\pi/64)\, G M^2 R^{-1}\,. \tag{6}$$

Lecture and Lab note on Plummer's model

Instructions to construct the phase space distribution are given in the Appendix

Virial Theorem?

Appendix: Generation of Initial Coordinates

Plummer's model was found to be convenient for a comparison of methods, and might be adopted as a standard model for such comparisons. Therefore we think it useful to give here a detailed prescription for the construction of initial positions and velocities.

There must be available a subroutine which generates normalized random numbers X, with uniform probability distribution between 0 and 1. We consider the system as defined by (1), (2) and (4), taking $G=1$, $M=1$, $R=1$ for convenience. In the equal-mass case, each star then has a mass $m=1/N$. From (1), the mass inside a sphere of radius r is

$$M(r)=r^3(1+r^2)^{-3/2}. \tag{A1}$$

In order to select a value of r for a star, we simply generate a random number X_1 and equate $M(r)$ to X_1, so that r is given by

$$r=(X_1^{-2/3}-1)^{-1/2}. \tag{A2}$$

The actual position (x, y, z) of the star should now be selected on the sphere of radius r, with uniform probability. This is done by the usual trick: we generate two normalized random numbers X_2 and X_3 and compute

$$\begin{aligned} &z=(1-2X_2)\,r, \quad x=(r^2-z^2)^{1/2}\cos 2\pi X_3, \\ &y=(r^2-z^2)^{1/2}\sin 2\pi X_3. \end{aligned} \tag{A3}$$

Next, we compute the velocity modulus for the same star. The maximum value of V at distance r from the centre is the escape velocity

$$V_e=(-2U)^{1/2}=2^{1/2}(1+r^2)^{-1/4}. \tag{A4}$$

We write $V/V_e=q$. Then (4) shows that the probability distribution of q is proportional to

$$g(q)=q^2(1-q^2)^{7/2}. \tag{A5}$$

A convenient way to sample q according to this distribution is provided by von Neumann's rejection technique. Possible values of q range from 0 to 1, and $g(q)$ is always less than 0.1. Therefore we generate two normalized random numbers X_4 and X_5; if $0.1\,X_5<g(X_4)$, we adopt $q=X_4$; if not, a new pair of random numbers is tried, until one is found which satisfies the inequality. The velocity modulus is then obtained, using (A4). Since the velocity distribution is isotropic, the three velocity coordinates u, v, w are computed from V in the same way as the three space coordinates from r, using two new random numbers X_6 and X_7:

$$\begin{aligned} &w=(1-2X_6)\,V, \quad u=(V^2-w^2)^{1/2}\cos 2\pi X_7, \\ &v=(V^2-w^2)^{1/2}\sin 2\pi X_7. \end{aligned} \tag{A6}$$

The whole procedure is repeated for each of the N stars. Finally, the values of m, x, y, z, u, v, w may be scaled to suit the numerical scheme used. If a cluster with mass M and energy $\mathscr{E}$ is desired, while keeping $G=1$, then masses should be multiplied by M, lengths by $(3\pi/64)\,M^2\,|\mathscr{E}|^{-1}$, and velocities by $(64/3\pi)\,|\mathscr{E}|^{1/2}\,M^{-1/2}$.

Assignment 2

https://github.com/jmduarte/phys141/blob/main/assignments/assignment2/assignment2.md

Problem 1: Plummer Sphere

(a) [50 points] Using your own code, generate the point distribution in (x, y, z, v_x, v_y, v_z) phase space for the Plummer sphere with 100,000 points as discussed in lecture and this lab note. The mass of the sphere is $M = 10^{11} M_\odot$ and the radius is $a = 1.5$ kpc. Note: the `mkplummer` app of Nemo is not acceptable as your solution.

(b) [20 points] Plot the corresponding theoretical mass density distribution as a function of the radial distance from the origin $P(r) = 4\pi r^2 \rho(r)$ and compare it with your point mass distribution. Use a histogram for the distribution.

Hint: You may need to normalize your distributions appropriately to compare them on the same scale.

(c) [20 points] Calculate the histogram of the energy $E = \frac{1}{2}v^2 + \Phi(r)$ distribution of the phase space points and compare it with the analytic expression $f \propto (-E)^{7/2}$.

(d) [241 only, 20 points] Show evidence that your velocity distribution agrees with the expected theoretical distribution at some value of r.

Hint: Consider points in thin spherical shell around radius r.

(e) [241 only, 20 points] Based on the Plummer model provided in lecture, prove Eq. (6) of this note, namely that the total energy is given by

$$\mathcal{E} = -\frac{3\pi G M^2}{64a}$$

Show that Eq. (6) satisfies the general virial theorem connecting the total energy, the kinetic energy, and potential energy of the phase space point distribution.