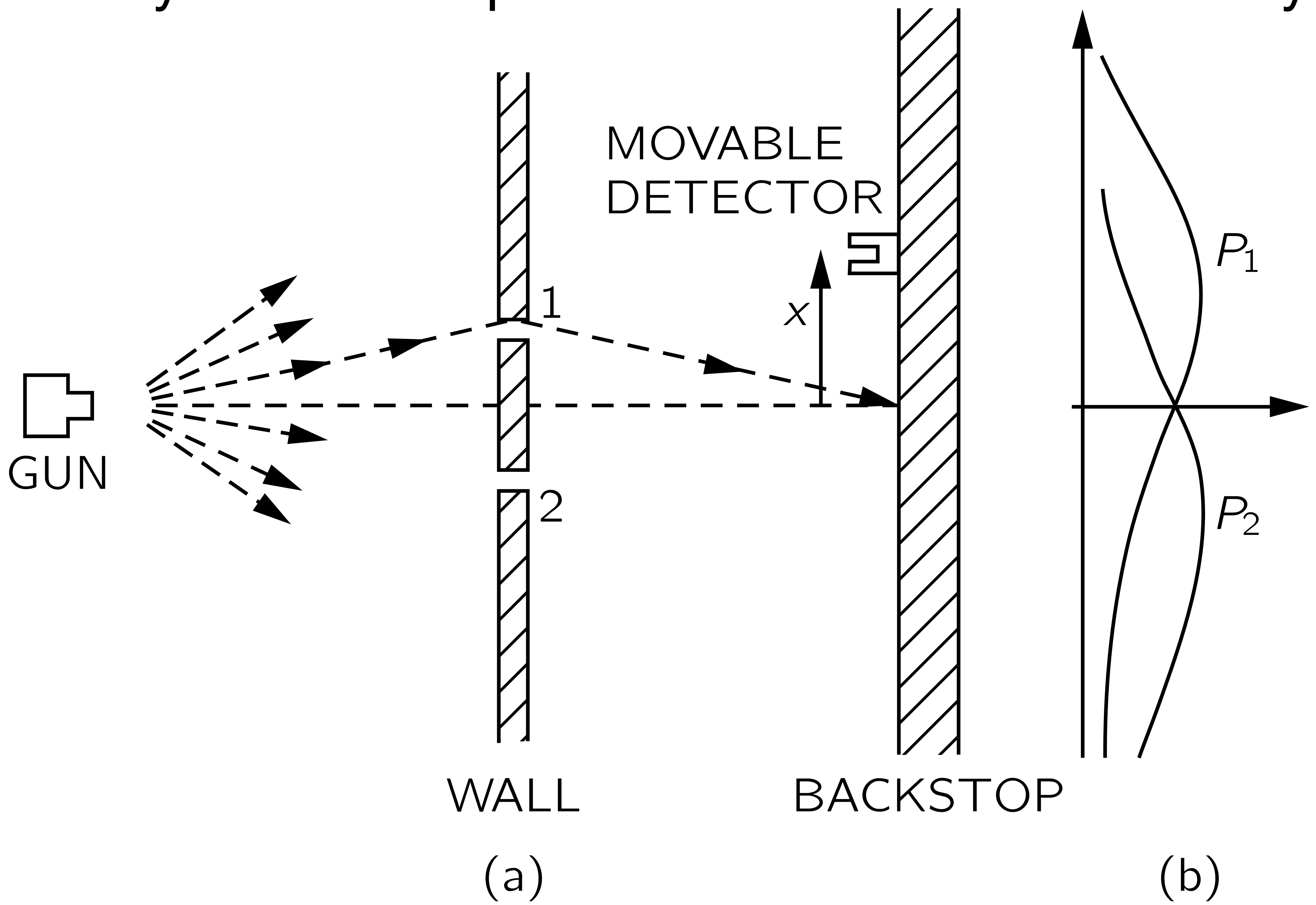
PHYS 142/242

Lecture 02: Feynman Path Integral

Javier Duarte — January 8, 2025

Double slit: darts

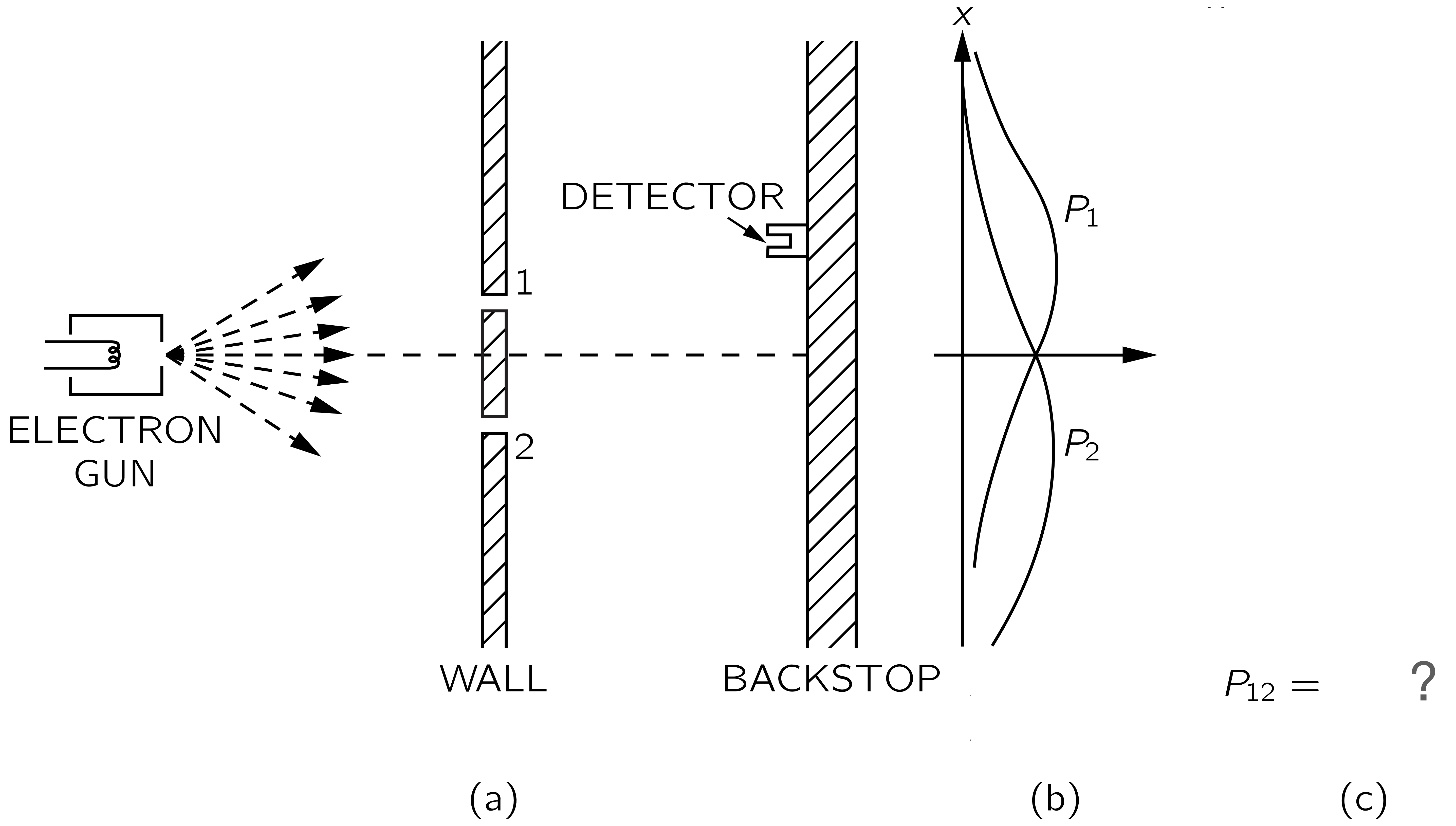
- Classical (macroscopic) behavior, e.g. a Nerf gun
- $P_i(x)$ is the probability a dart hits position x on the screen if only hole i is open



- What is the probability distribution $P_{12}(x)$ if both holes are open?
- Each dart that travels from the Nerf gun to the backstop must go through either hole 1 or 2
- The probability of arrival at x is the sum of two parts: $P_1(x)$ the probability of arrival passing through hole 1, plus $P_2(x)$, the probability of arrival passing through hole 2

Double slit: electrons

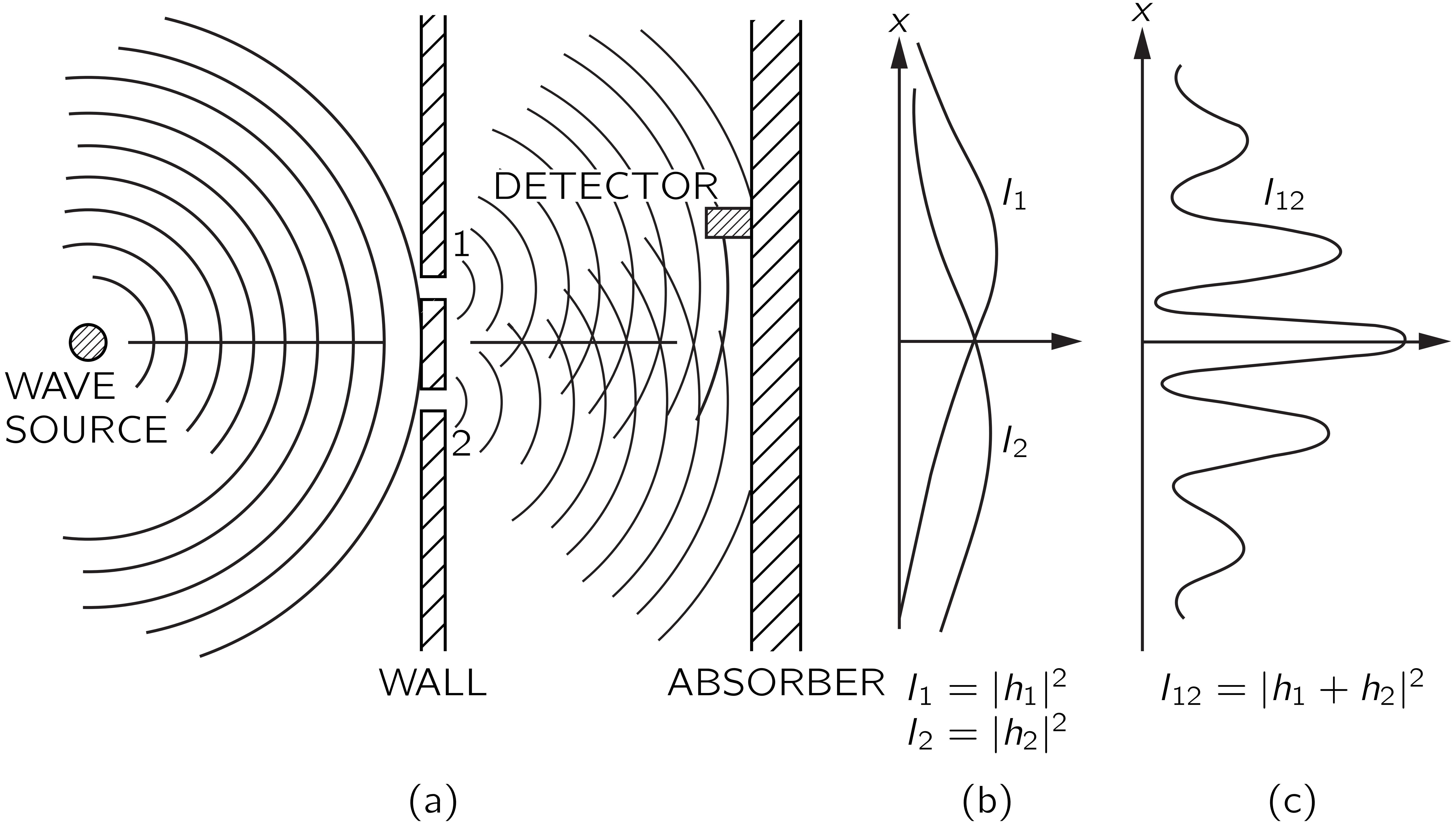
- How do individual electrons behave?



- If we perform the experiment, we observe the following pattern
- Probability amplitude $\phi_i(x)$ for each hole i, and the probability amplitudes sum $\phi_{12}(x) = \phi_1(x) + \phi_2(x)$; interpret ***intensity*** as probability $P_{12}(x) = |\phi_{12}(x)|^2$

Double slit: waves

- This is a familiar phenomenon! Electrons behave like waves

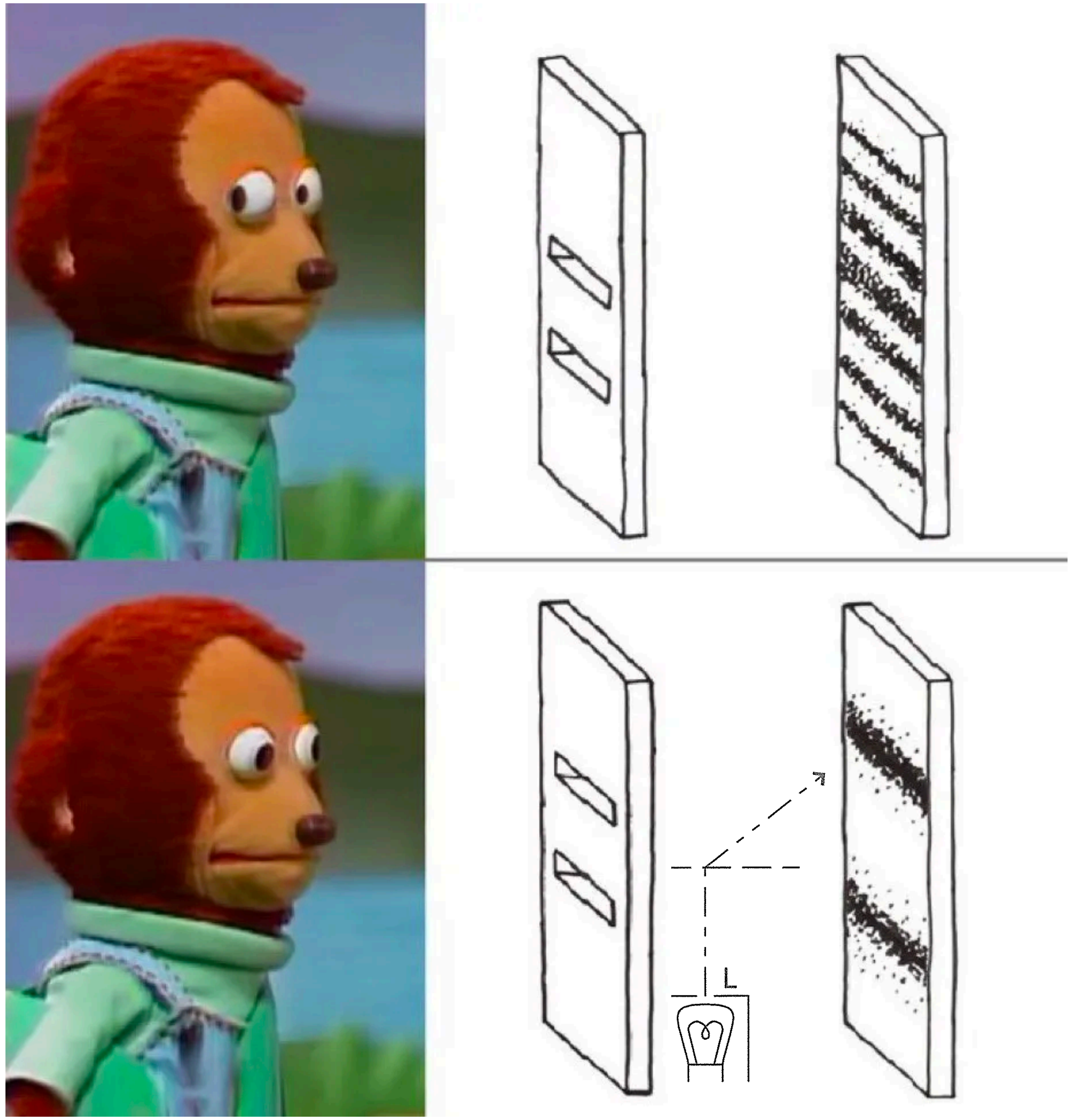


Double slit: meaning

- What does this mean?
  - When both holes are open, ***it is not true that the electron goes through either one hole or the other***
  - ***Instead the two alternatives "interfere"***

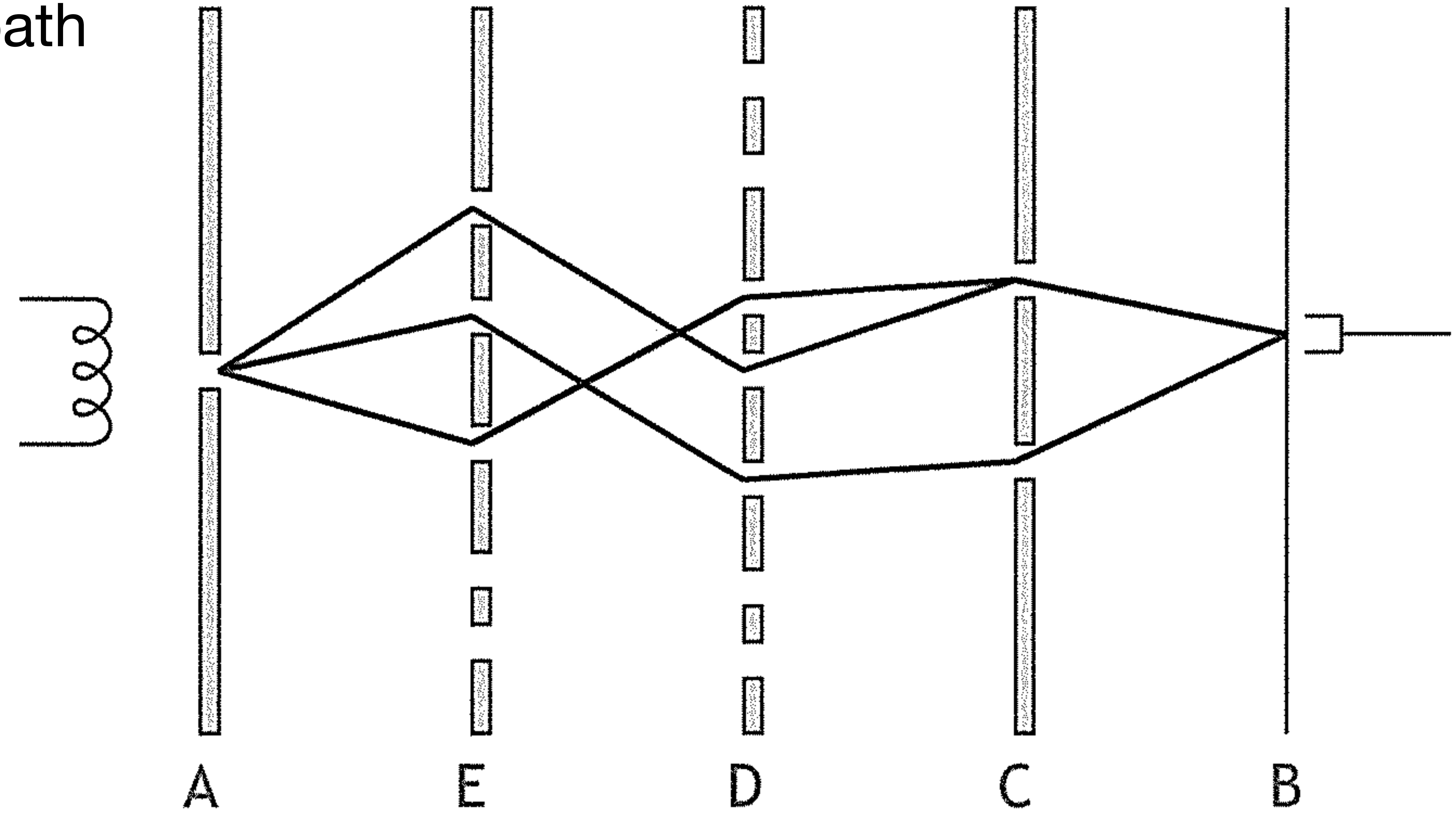
Double slit: effect of observation

- What if we put some kind of detector to tell for sure which hole the electron passes through?
- It destroys the interference pattern! Left with the classical behavior
- Just by watching the electrons, we change the probability that they arrive at x
- How is this possible?
    - Detection implies interaction with the electron, e.g. scattering with a photon, which alters its motion and its probability of arrival at x



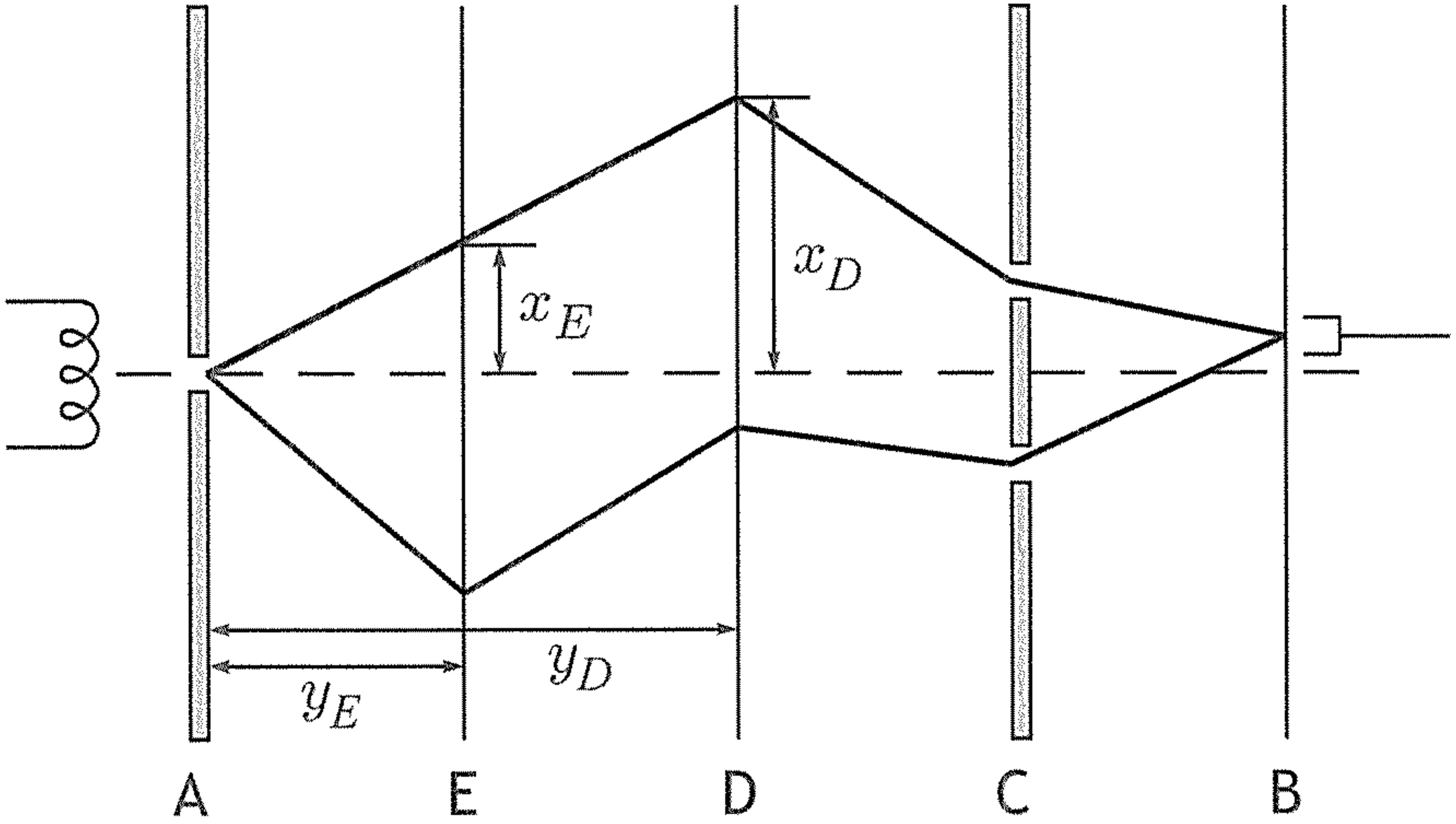
From double slit to Feynman path integral

- Consider adding screens E and D with several holes
- For each route, there is an amplitude
- When all the holes are open we must sum all these amplitudes, one for each possible path



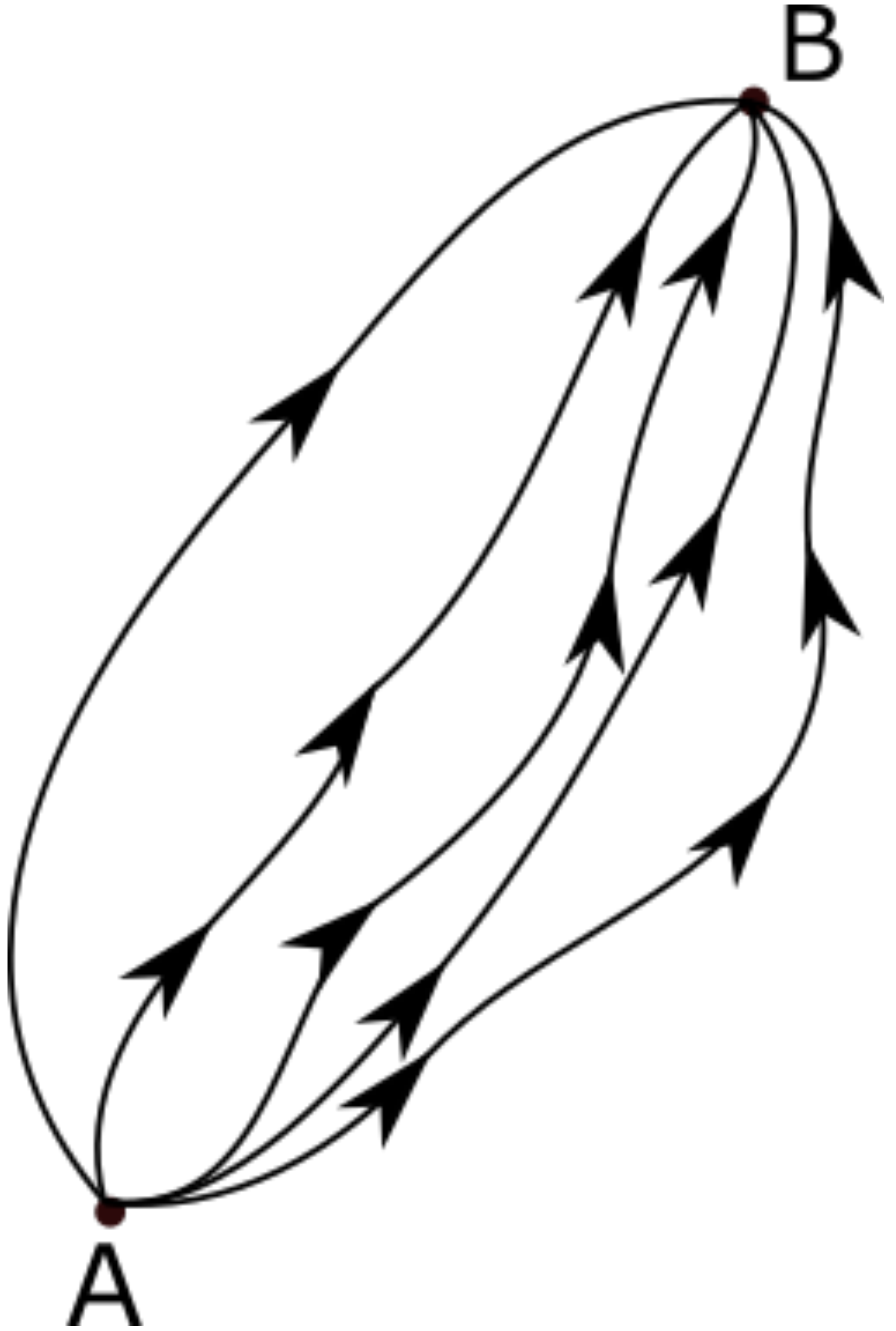
From double slit to Feynman path integral

- More and more holes are cut in the screens at E and D until eventually, the electron has a continuous range of positions it can pass through
- The sum becomes a double integral over x_E and x_D



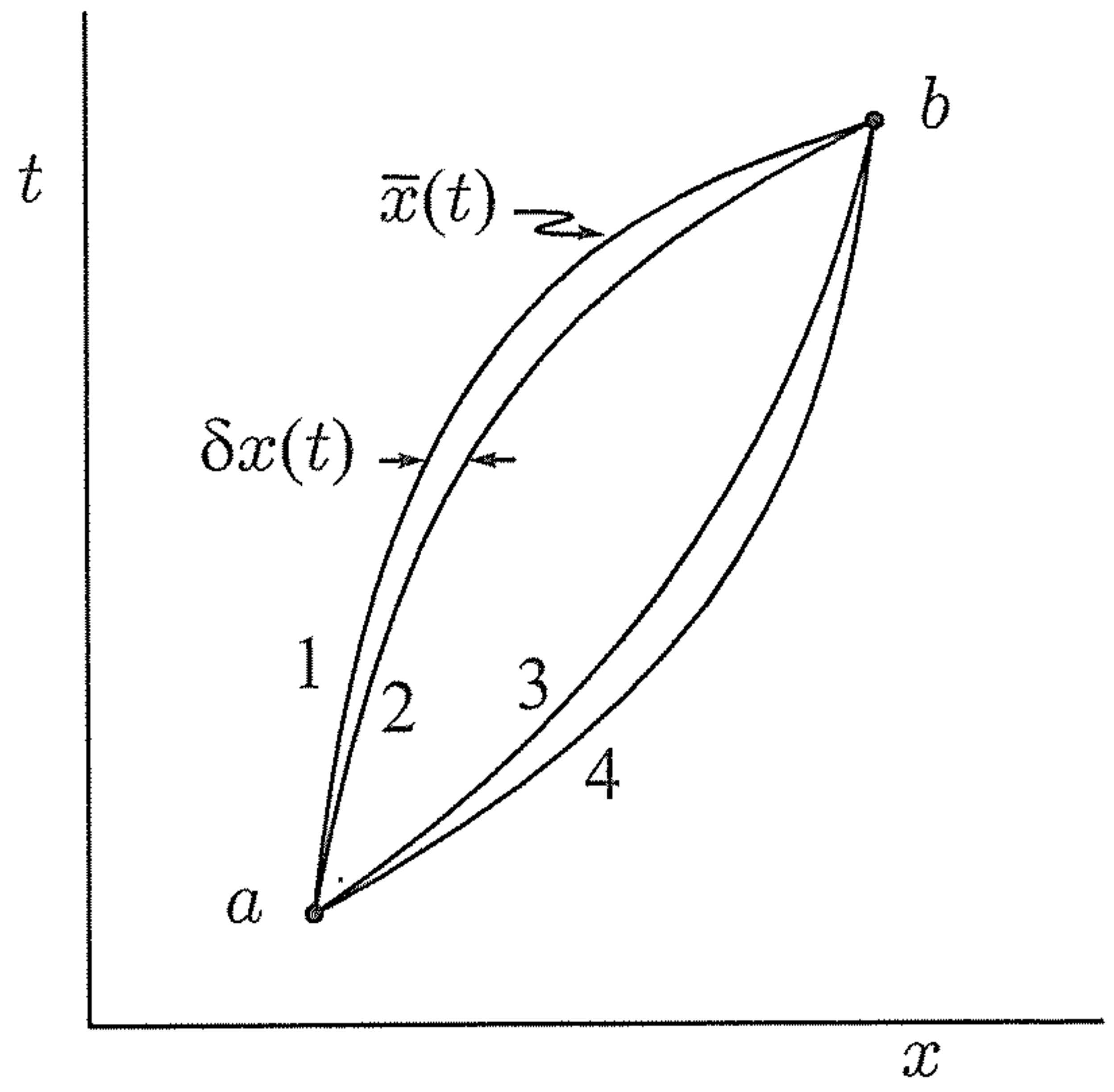
Feynman path integral

- In quantum mechanics, the amplitude to go from A to B is the sum of contributions $\phi[(x(t)]$ from each path
- $$K(B, A) = \sum_{\text{paths from A to B}} \phi[x(t)]$$
- But what does each contribution look like?



Lagrangian mechanics reminder

- Define the Lagrangian $L(x, \dot{x}, t) \equiv K(x, \dot{x}) - V(x, t)$
  - Example: free particle, $K = \frac{1}{2}m\dot{x}^2$, $V = 0$
  - Example: harmonic oscillator, $K = \frac{1}{2}m\dot{x}^2$, $V = \frac{1}{2}m\omega^2 x^2$



- Action $S = \int_{t_a}^{t_b} L(x, \dot{x}, t)dt$
- The principle of least action (PLA): The classical path $\bar{x}(t)$ is that for which S is an extremum; S is unchanged (to first order) if the path $\bar{x}(t)$ is modified slightly
  - If we vary $\bar{x}(t)$ by a small amount $\delta x(t)$
  - Boundary condition: $\delta x(t_a) = \delta x(t_b) = 0$
  - $\delta S = S[\bar{x} + \delta x] - S[\bar{x}] = 0$ (to first order in δx)

Lagrangian mechanics reminder

$$S[\overline{x}+\delta x] = \int_{t_a}^{t_b} L(\dot{x}+\delta\dot{x}, x+\delta x, t)dt = \int_{t_a}^{t_b} \left[L(\dot{x}, x, t) + \delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x\frac{\partial L}{\partial x}\right] dt$$

$$= S[x] + \int_{t_a}^{t_b} \left[\delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x\frac{\partial L}{\partial x}\right] dt$$

Using integration by parts,

$$\delta S = 0 = \int_{t_a}^{t_b} \left[\delta\dot{x}\frac{\partial L}{\partial \dot{x}} + \delta x\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \delta x\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \delta x\frac{\partial L}{\partial x}\right] dt$$

Euler-Lagrange equations

$$= \int_{t_a}^{t_b} \left[\frac{d}{dt}\left(\delta x\frac{\partial L}{\partial \dot{x}}\right) - \delta x\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \delta x\frac{\partial L}{\partial x}\right] = \underbrace{\left[\delta x\frac{\partial L}{\partial \dot{x}}\right]_{t_a}^{t_b}}_{=0} - \int_{t_a}^{t_b} \delta x \underbrace{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}\right]}_{=0} dt$$

because $\delta x(t_a) = \delta x(t_b) = 0$

Euler-Lagrange and Newton

Example: $L = \frac{1}{2}m\dot{x}^2 - V(x)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt}(m\dot{x}) = m\ddot{x} = -\frac{dV}{dx}$$

Equivalent to Newton's 2nd law of motion

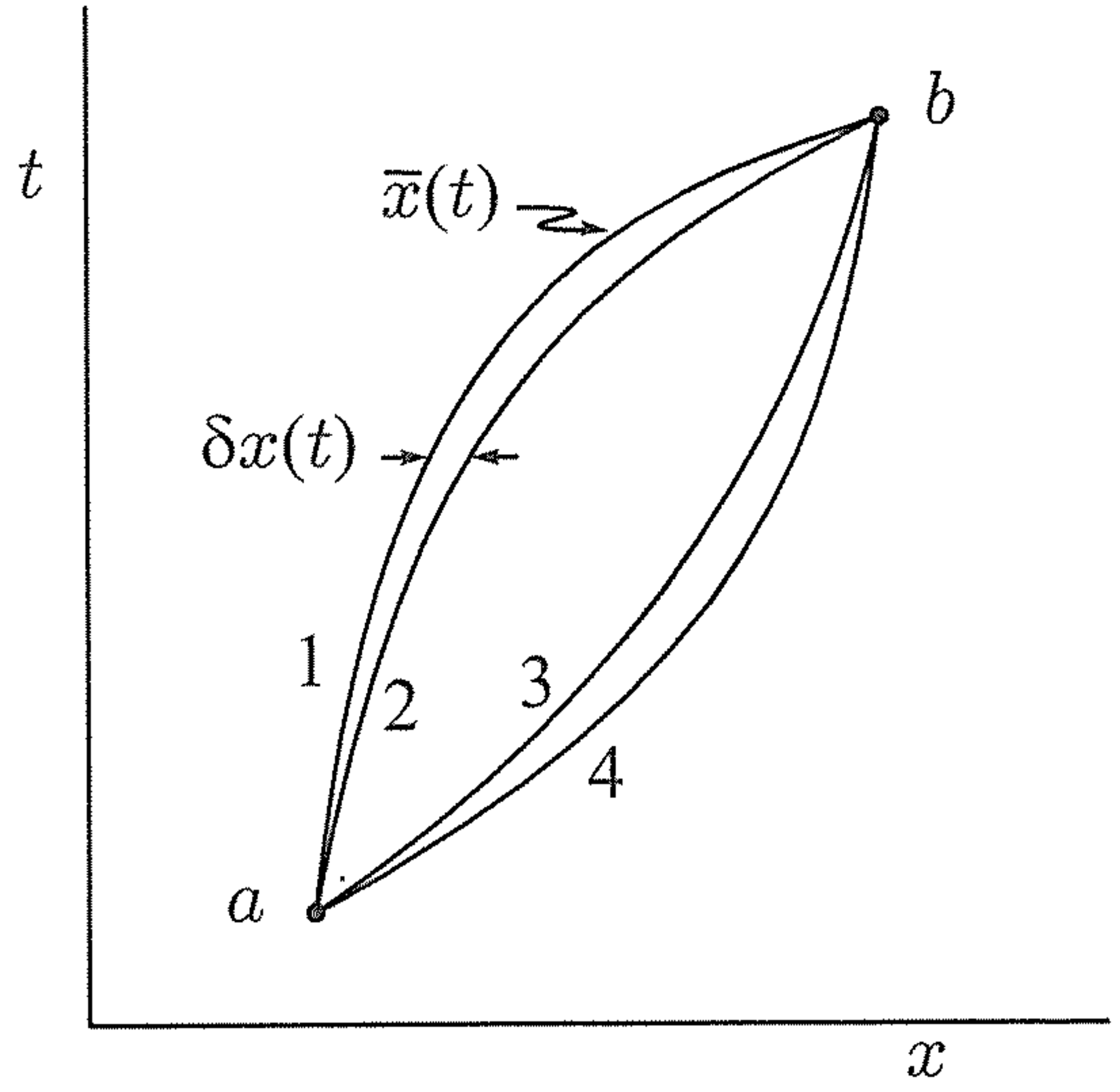
$$F \equiv -\frac{dV}{dx} = ma$$

Classical vs. Quantum

- In classical mechanics, the form of the action integral $S = \int L dt$ is interesting, moreso than the extreme value S_{cl}
    - Leads to the ***Euler-Lagrange equations*** $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = \dfrac{\partial L}{\partial x}$
    - Solving these determines the ***path of least action***
- In quantum mechanics, both the form of the integral and the extreme value are important
    - All paths contribute (not just the extremal path)

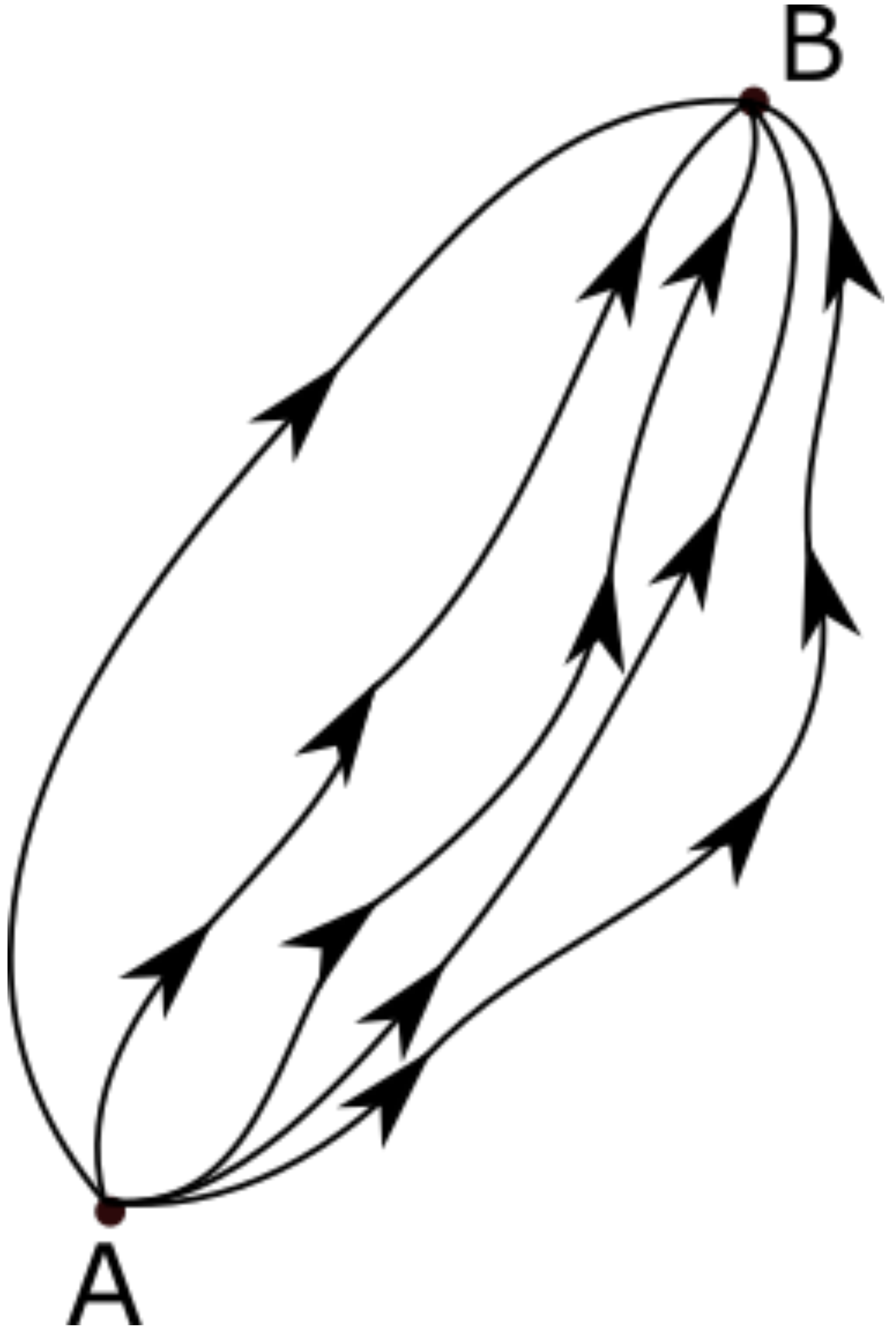
Example: Classical action for free particle

- In Quiz 1, you will show the classical action for a free particle of mass m is
$$S = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}$$
- What are the units?
  - cgs: $\mathrm{cm^2\, g/s}$
  - Particle physics units: $\mathrm{MeV\ s}$



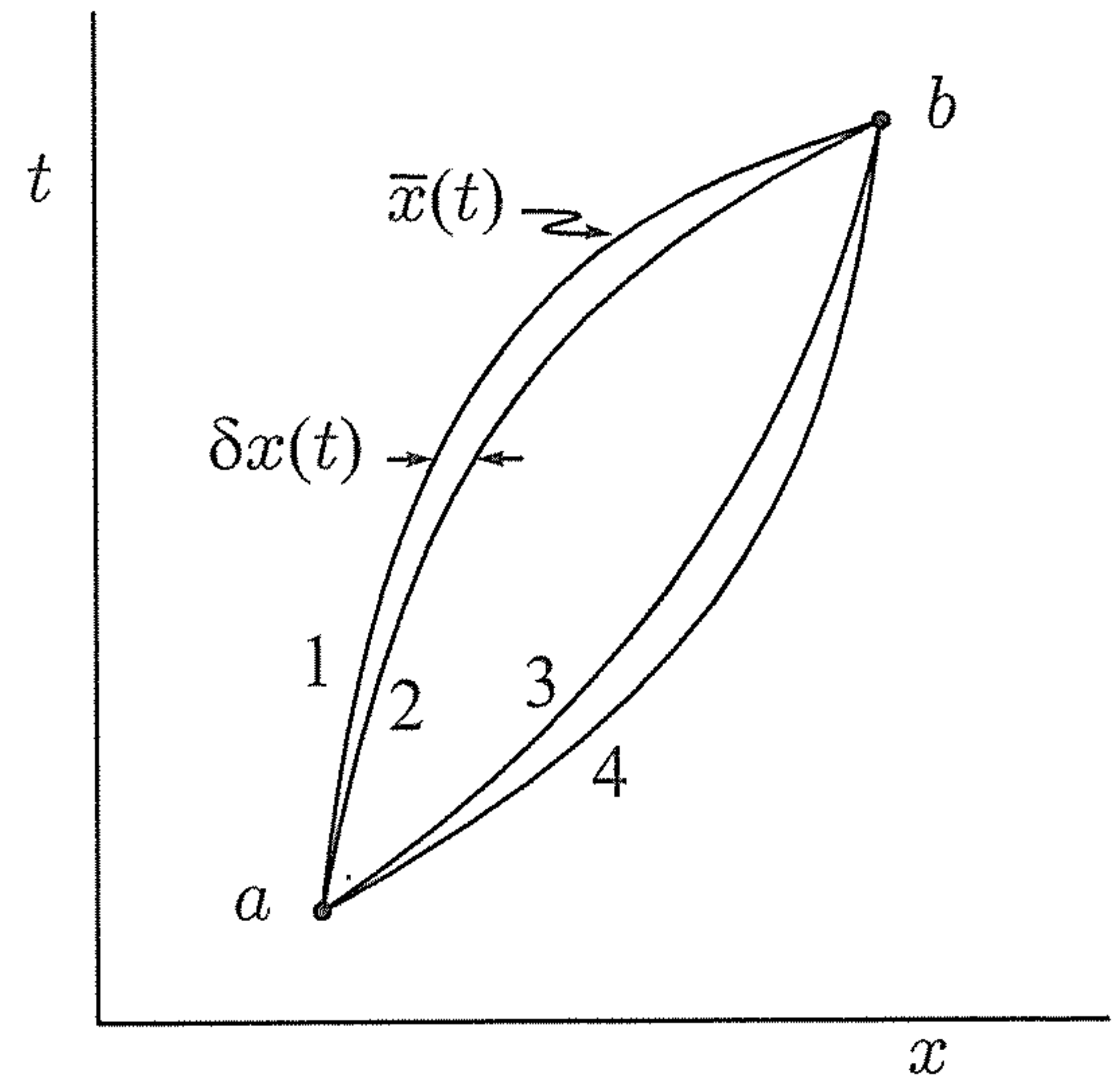
Feynman path integral

- In quantum mechanics, the amplitude to go from A to B is the sum of contributions $\phi[(x(t)]$ from each path
- $$K(B,A) = \sum_{\text{paths from A to B}} \phi[x(t)]$$
- The contribution of a path has a phase proportional to the *classical action* S
    - $\phi[x(t)] = (\text{const}) e^{(i/\hbar)S[x(t)]}$
- This is a generalization of the classical principle of least action, sometimes called the ***quantum action principle***
    - Contains the classical principle in the limit $\hbar \to 0$ (or equivalently when $S \gg \hbar$



Classical limit

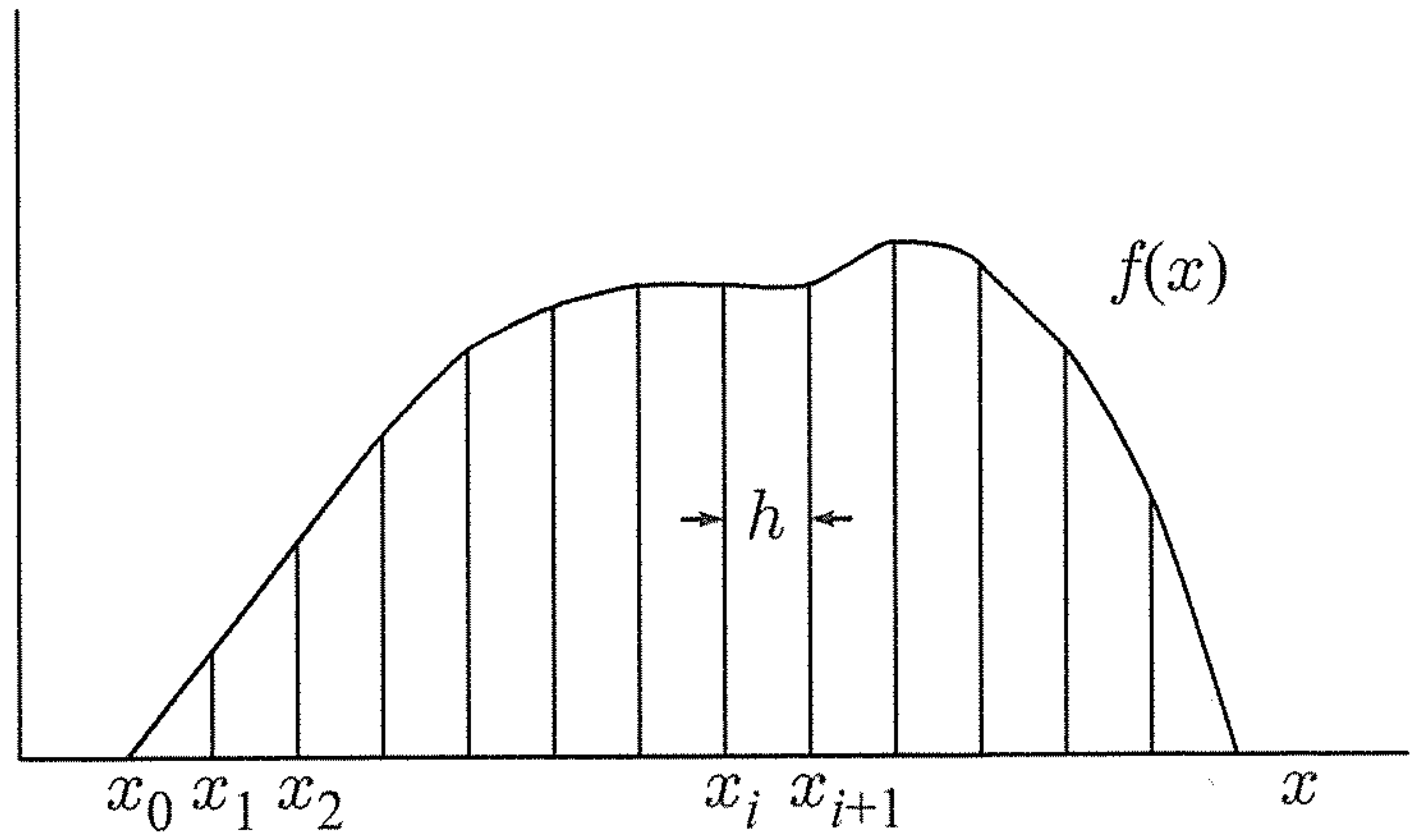
- The phase of each contribution is $e^{iS/\hbar} = \cos(S/\hbar) + i\sin(S/\hbar)$
  - Reminder: $\hbar = 6.6261 \times 10^{-27}\,\mathrm{cm}^2\,\mathrm{g/s}$ (tiny!)
- Near the classical path $\bar{x}(t)$, S varies very little
  - e.g. 1 and 2 contribute with the same phase and ***constructively interfere***
- Far from the classical path, S varies a lot in units of $\hbar$
  - e.g. 3 and 4 contribute with different phases and ***destructively interfere***



- Only paths in the vicinity of $\bar{x}(t)$ have important contributions (that don't cancel), and ***in the classical limit we only need to consider this trajectory***
  - Classical laws of motion arise from the quantum laws!

Defining the path integral

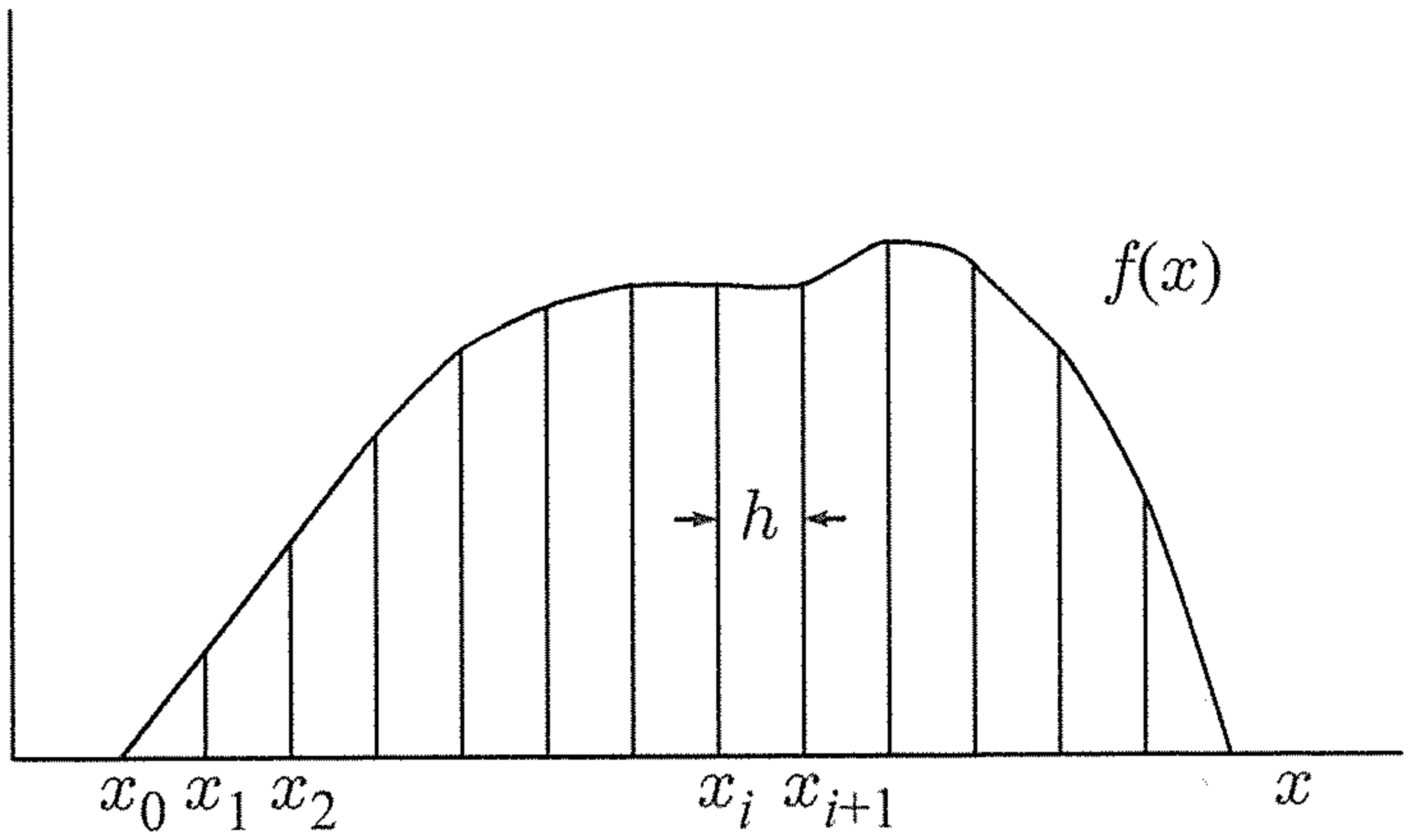
- How do we construct the path integral/sum?
- $$K(x_B, t_B; x_A, t_A) = (\text{const}) \sum_{\text{all paths}} e^{\frac{i}{\hbar} S_{\text{path}}}$$
  - Note the probability amplitude for the particle to travel from position x_A at time t_A to position x_B at time t_B is sometimes called the **kernel** or the **propagator**
- By analogy with the Riemann integral/sum



Riemann sum

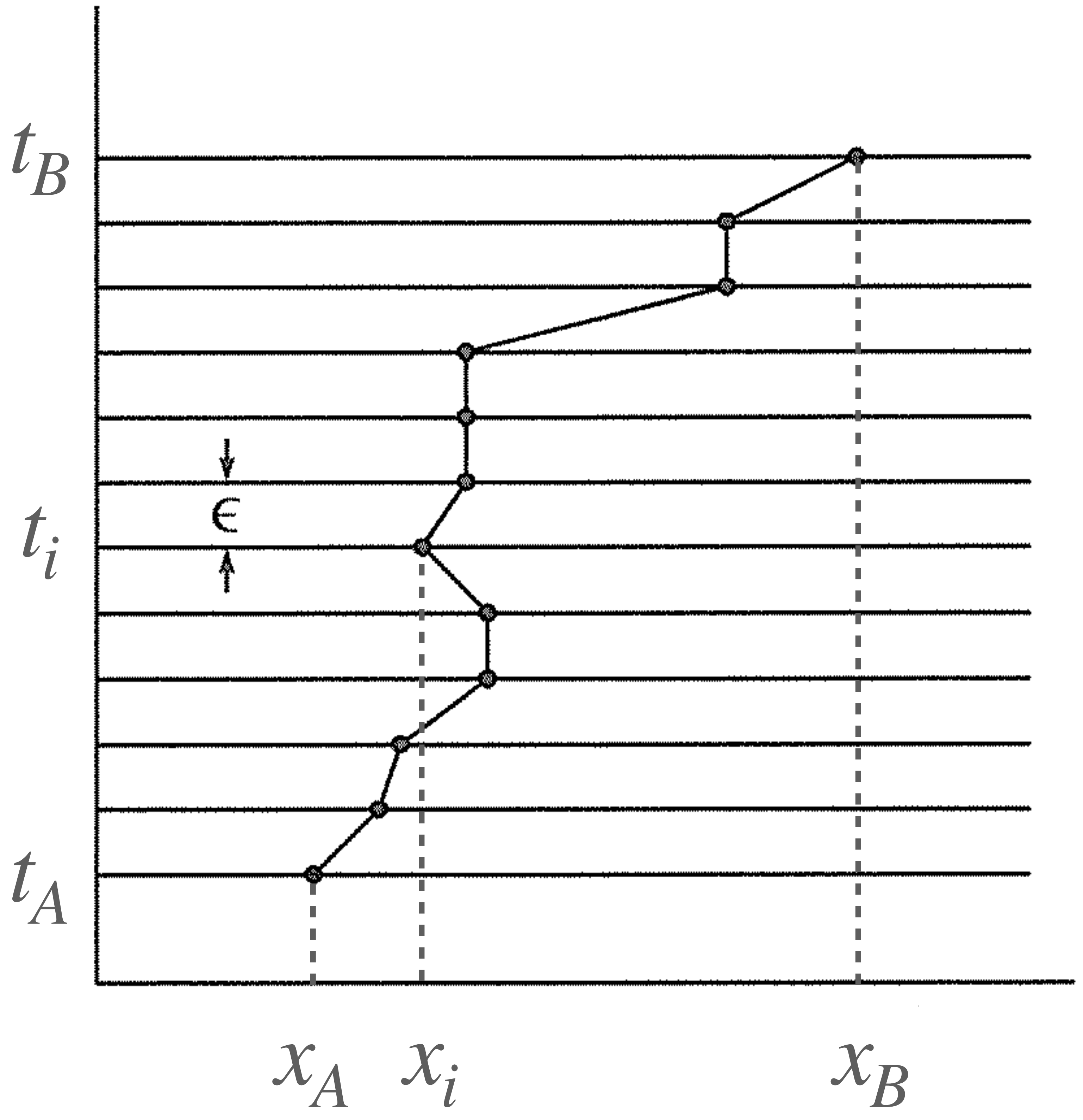
- Define N equally spaced x coordinates (separation h) as a representative subset of all coordinates
- Area under the curve A is the sum of all function values evaluated at each x_i, multiplied by an overall normalization factor h,

$$A = \lim_{h \to 0, N \to \infty} h \sum_{i=0}^{N-1} f(x_i)$$



Defining the path integral

- First, choose a subset of all paths from (x_A, t_A) to (x_B, t_B) as follows
- Divide time into steps of width ϵ
- At each time t_i, the path passes through some chosen point x_i
- We construct a path by connecting all the points with straight lines (i.e. constant velocity)



- Summary:

$$N\epsilon = t_B - t_A, \qquad \epsilon = t_{i+1} - t_i, \qquad t_0 = t_A, \qquad t_N = t_B$$

$$x_0 = x_A, \qquad x_N = x_B$$

Defining the path integral

- Define the sum over all such paths by taking a multiple integral over all x_i coordinate choices
  - $K(x_B, t_B; x_A, t_A) \sim \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} dx_1 dx_2 \cdots dx_{N-1}$
  - Note: to get the limit to exist we need to know the normalization factor
- For now, we'll take the factor for granted, but we will show how it's derived later
- $K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0,\ N \to \infty} \frac{1}{C} \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} \frac{dx_1}{C} \frac{dx_2}{C} \cdots \frac{dx_{N-1}}{C}$
  - $C = \left(\frac{2\pi\hbar\epsilon}{m} \right)^{\frac{1}{2}}$; Overall normalizing factor is C^{-N}
  - $S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$