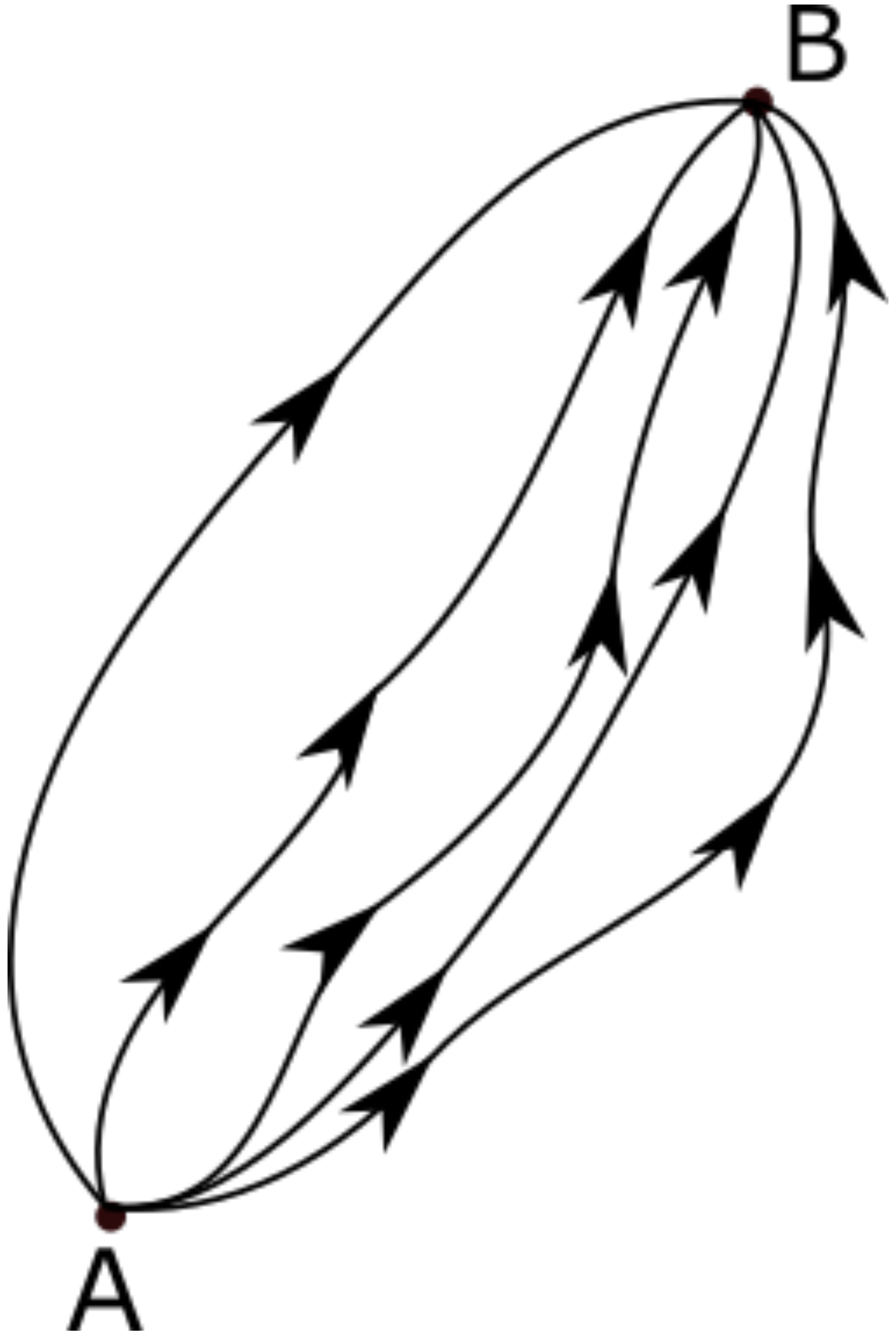
PHYS 142/242

Lecture 02: Feynman Path Integral (Continued)

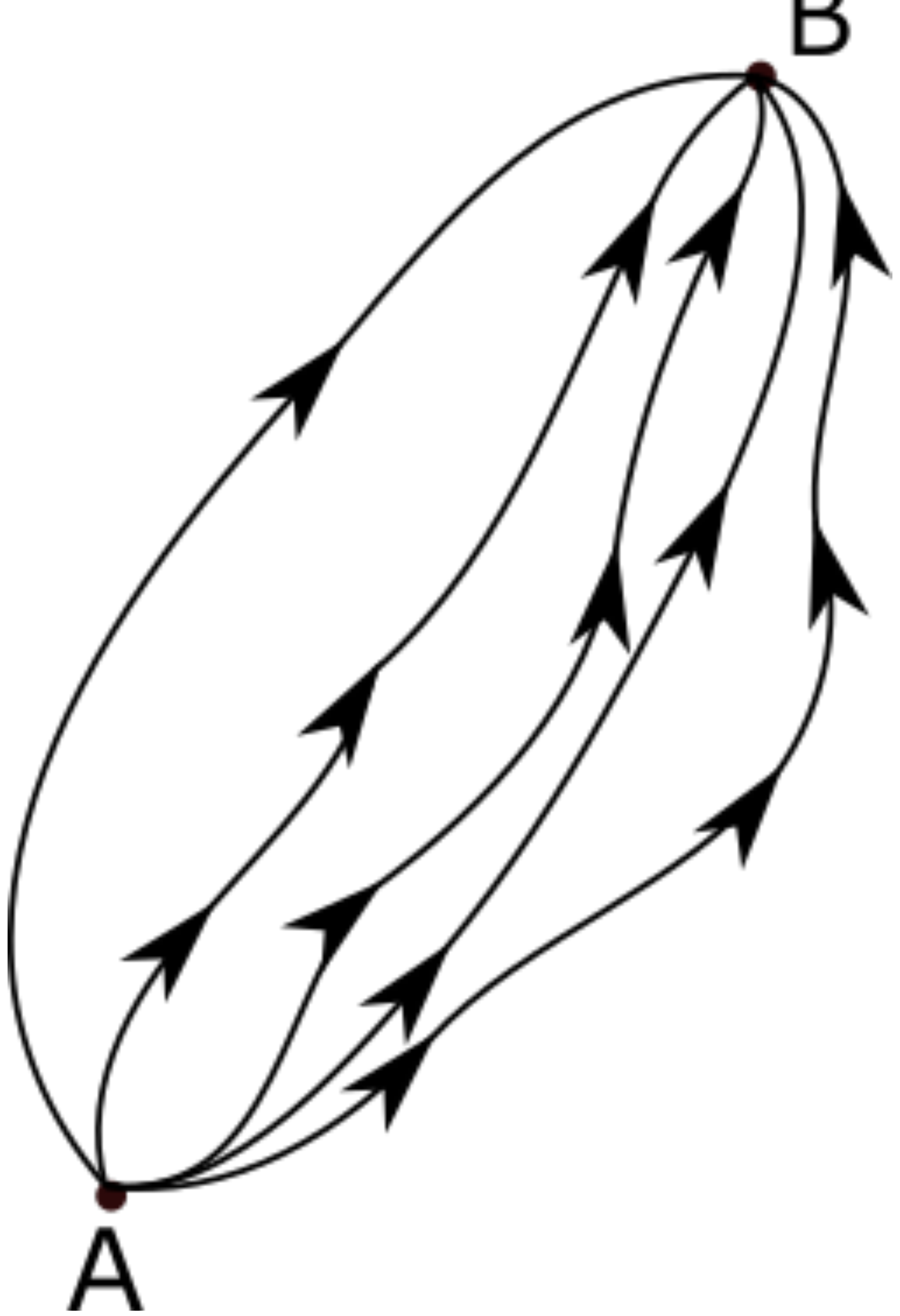
Javier Duarte — January 12, 2024

Feynman path integral



Feynman path integral

- In quantum mechanics, the amplitude to go from A to B is the sum of contributions $\phi[(x(t)]$ from each path
- $$K(B,A) = \sum_{\text{paths from A to B}} \phi[x(t)]$$



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- The contribution of a path has a phase proportional to the *classical action* $S = \int L dt$
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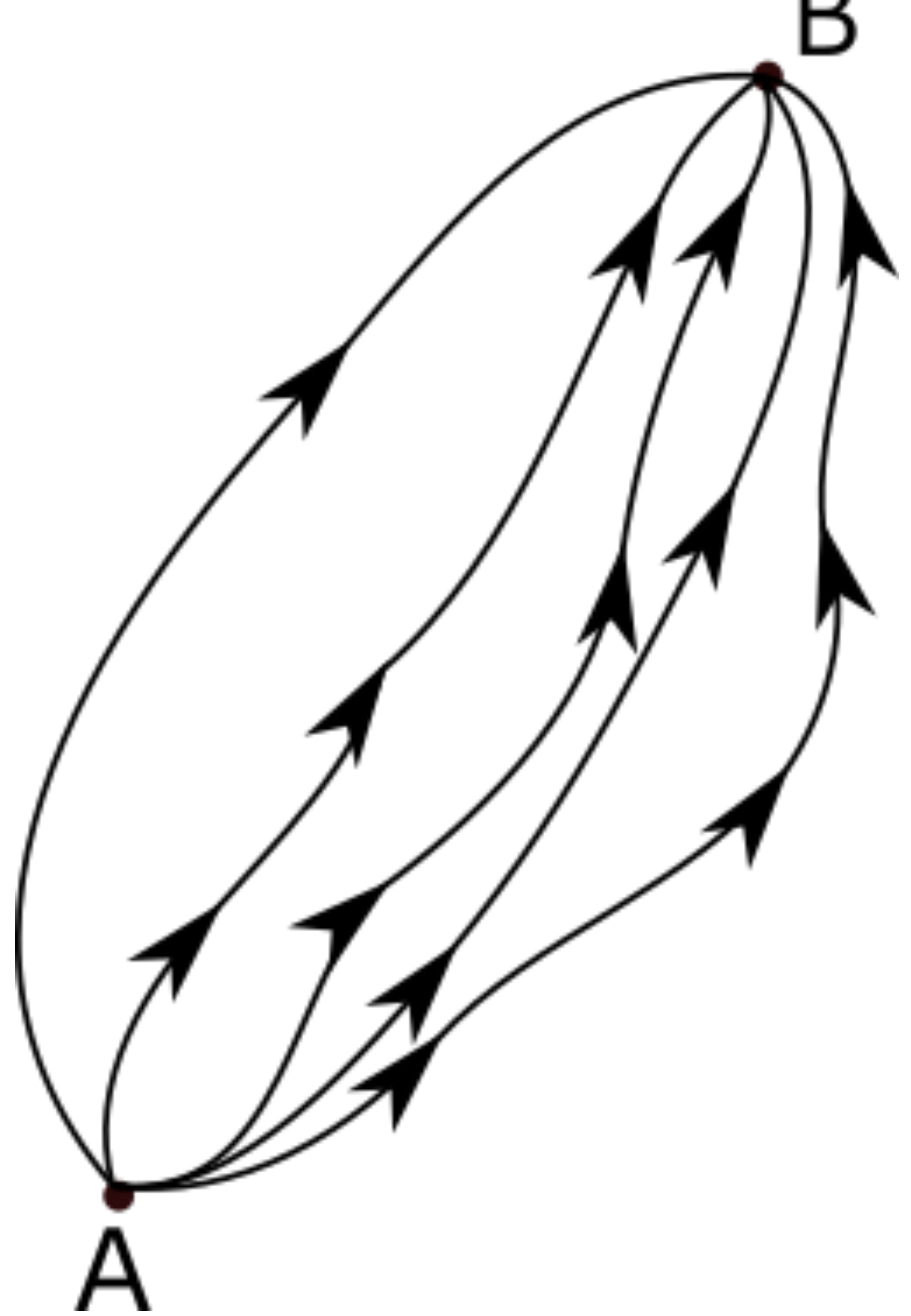

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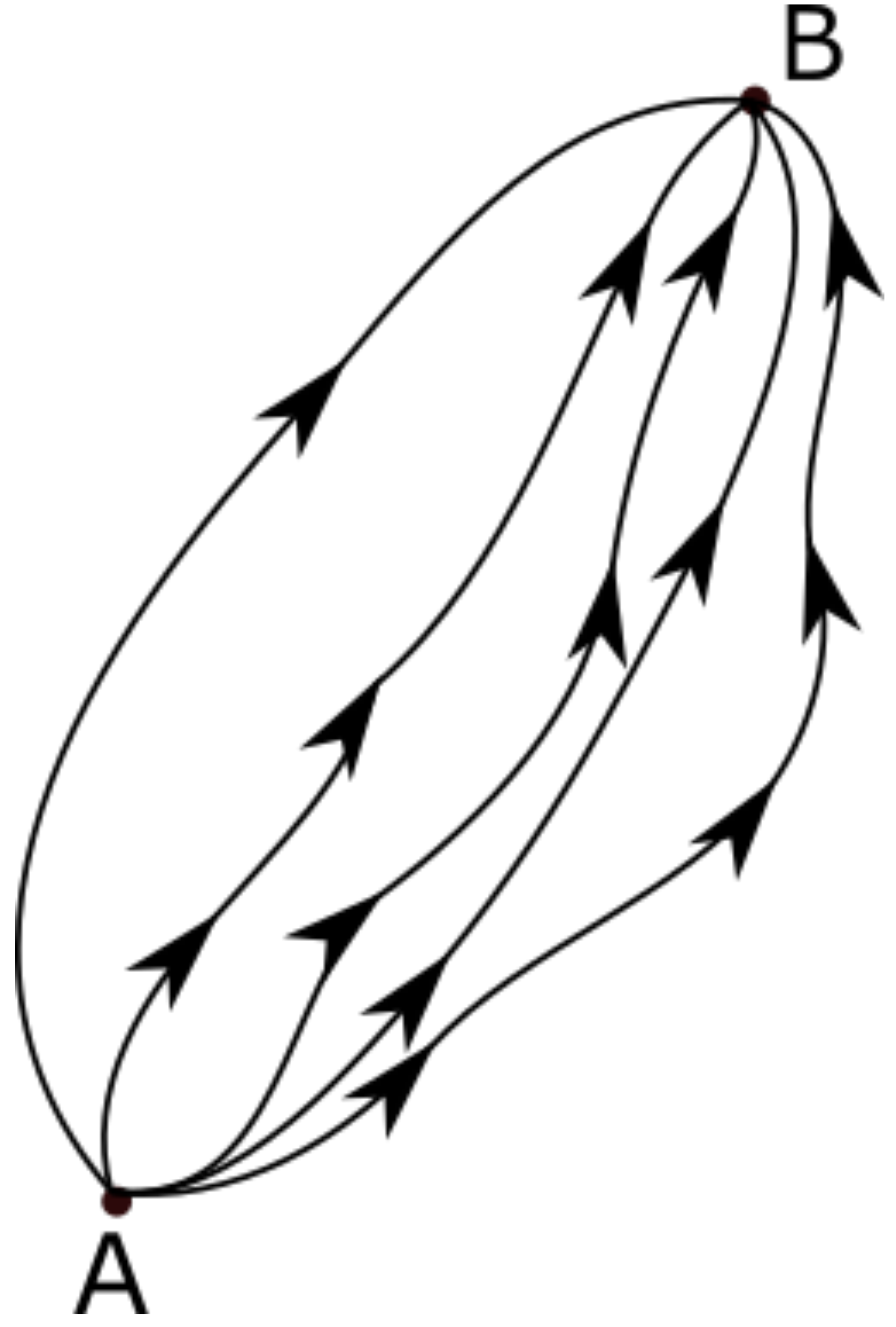
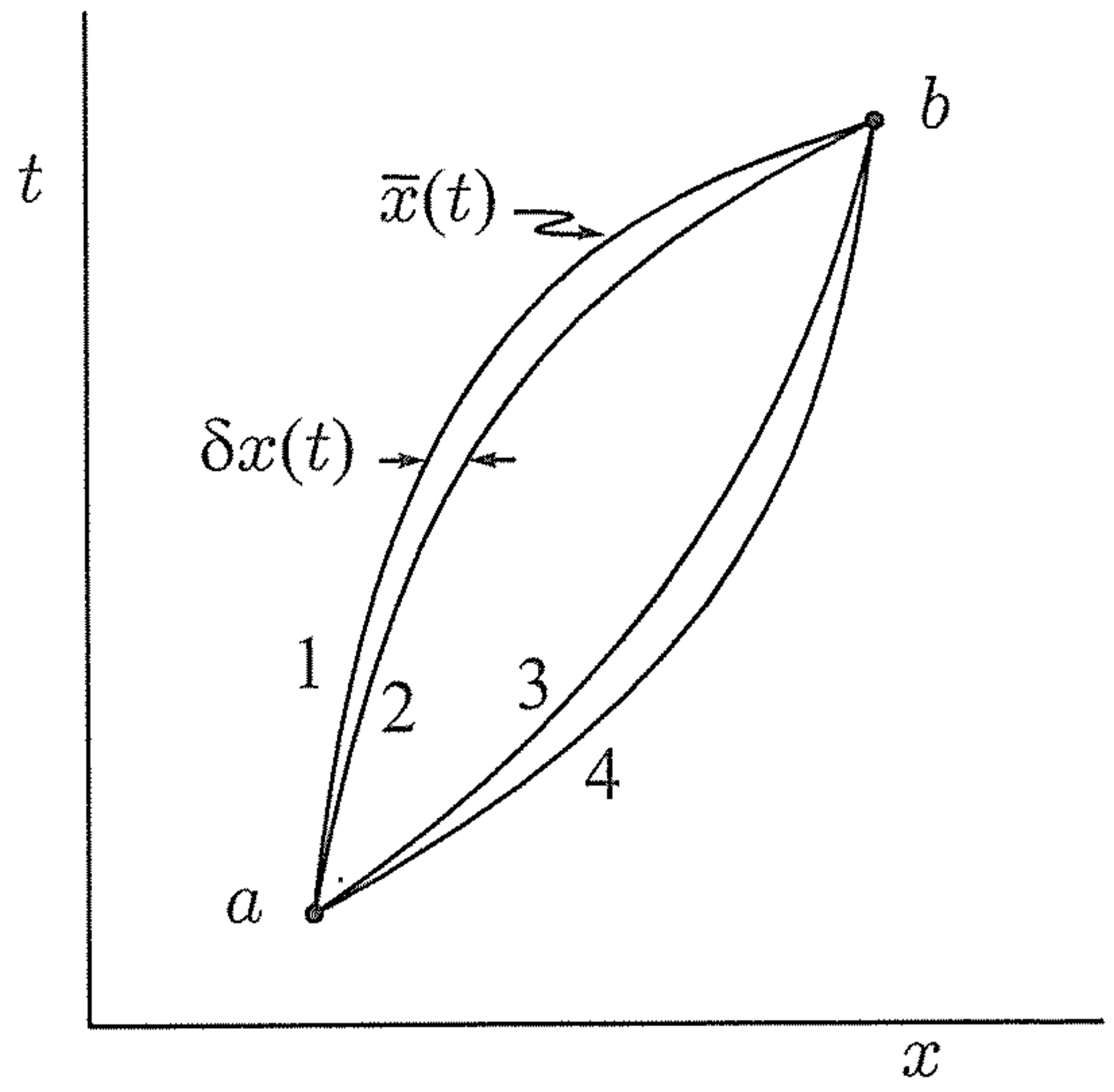
  - $\phi[x(t)] = (\text{const}) e^{(i/\hbar)S[x(t)]}$
- This is a generalization of the classical principle of least action, sometimes called the ***quantum action principle***
  - Contains the classical principle in the limit $\hbar \to 0$ (or equivalently when $S \gg \hbar$

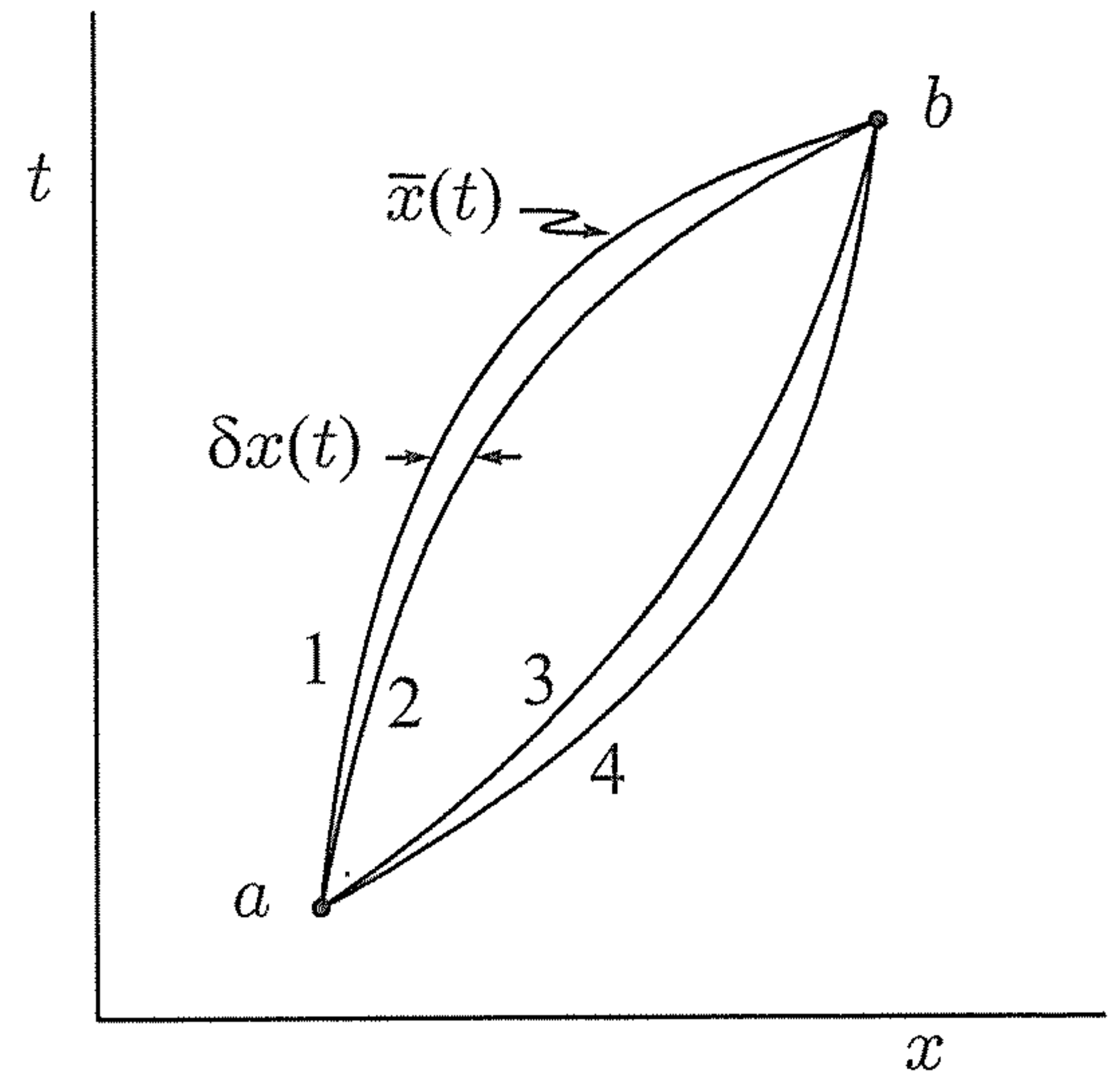


Classical limit



Classical limit

- The phase of each contribution is $e^{iS/\hbar} = \cos(S/\hbar) + i\sin(S/\hbar)$
  - Reminder: $\hbar = 6.6261 \times 10^{-27}\,\mathrm{cm}^2\,\mathrm{g/s}$ (tiny!)



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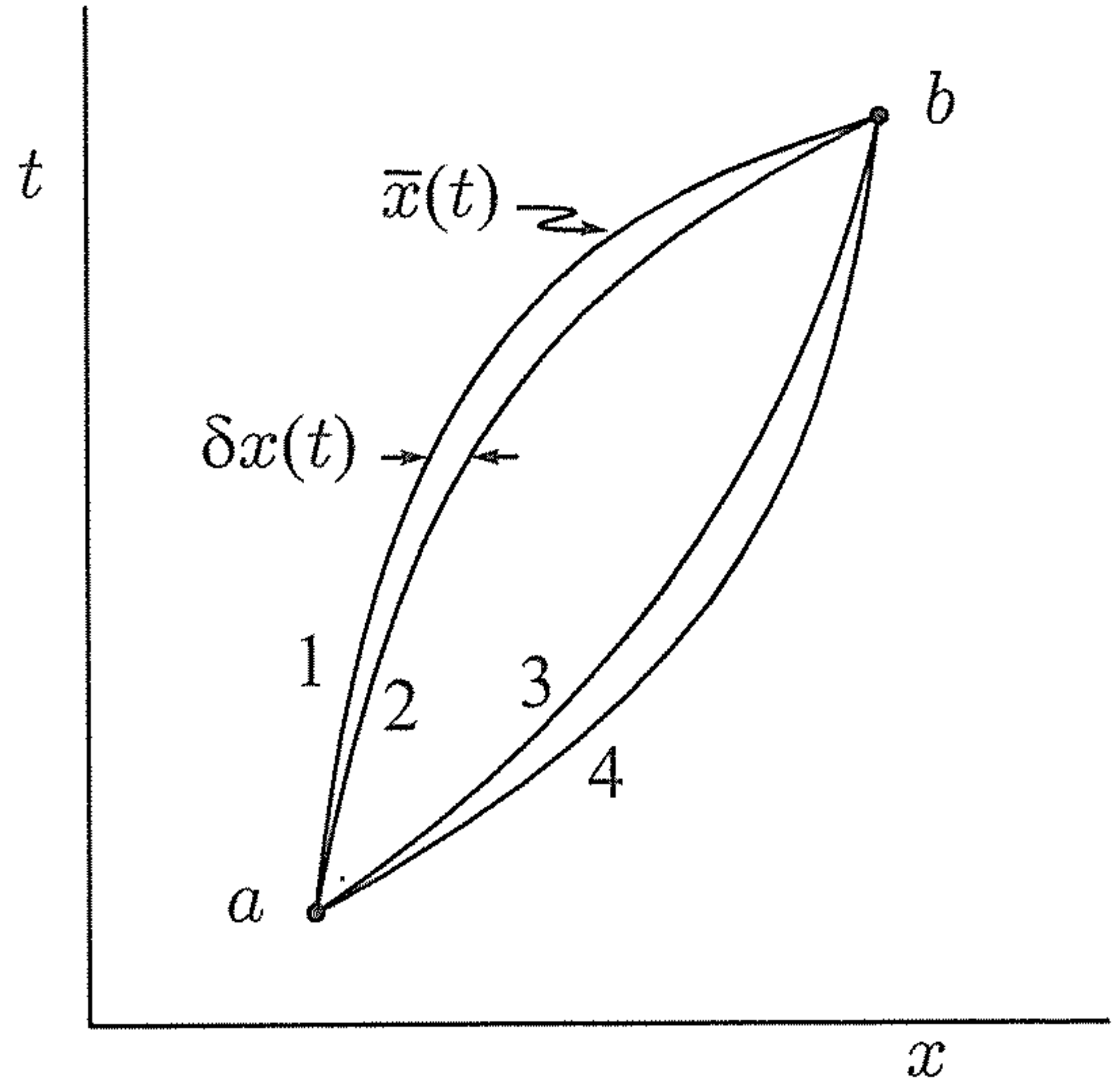
    - e.g. 1 and 2 contribute with the same phase and ***constructively interfere***



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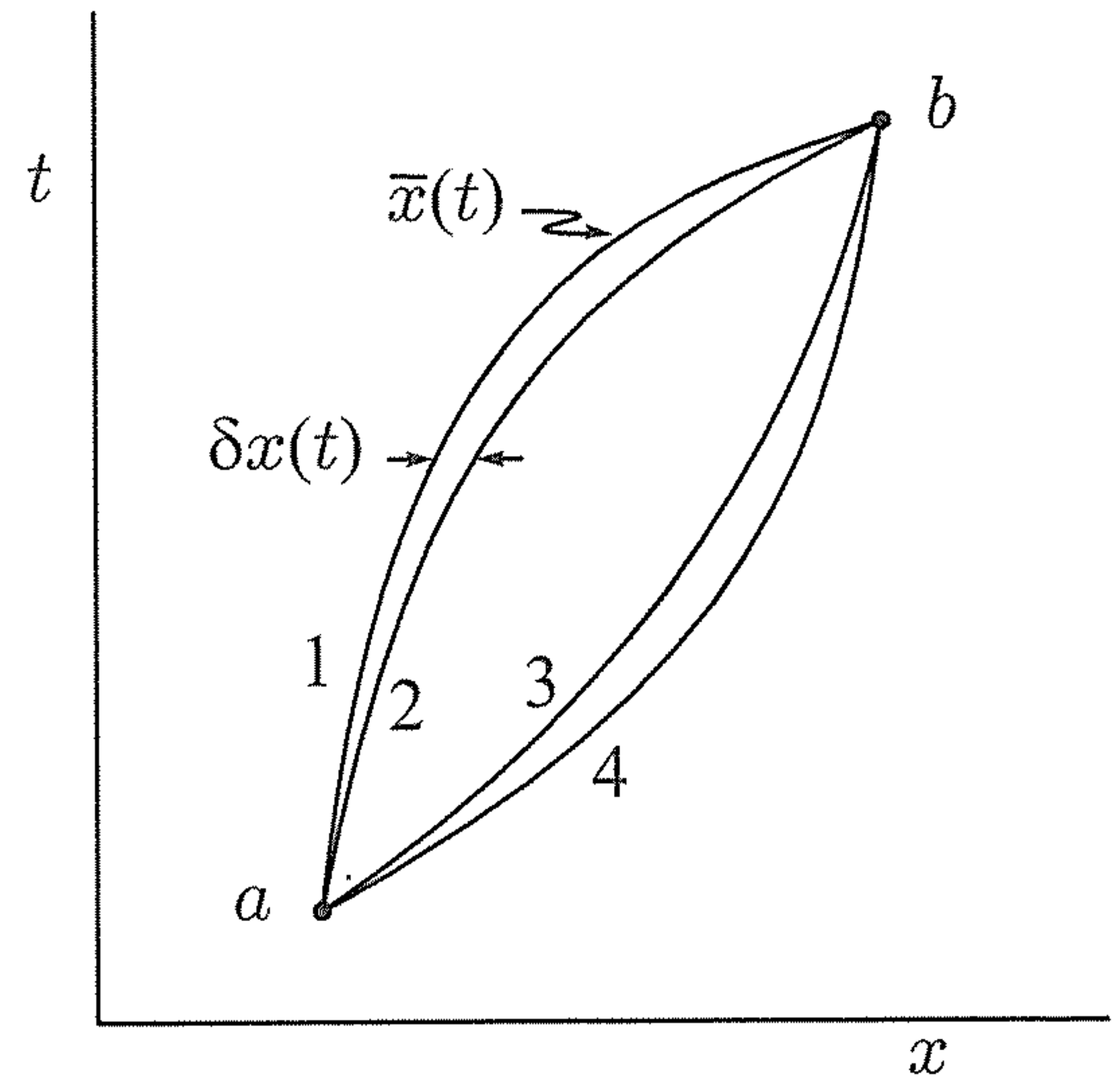

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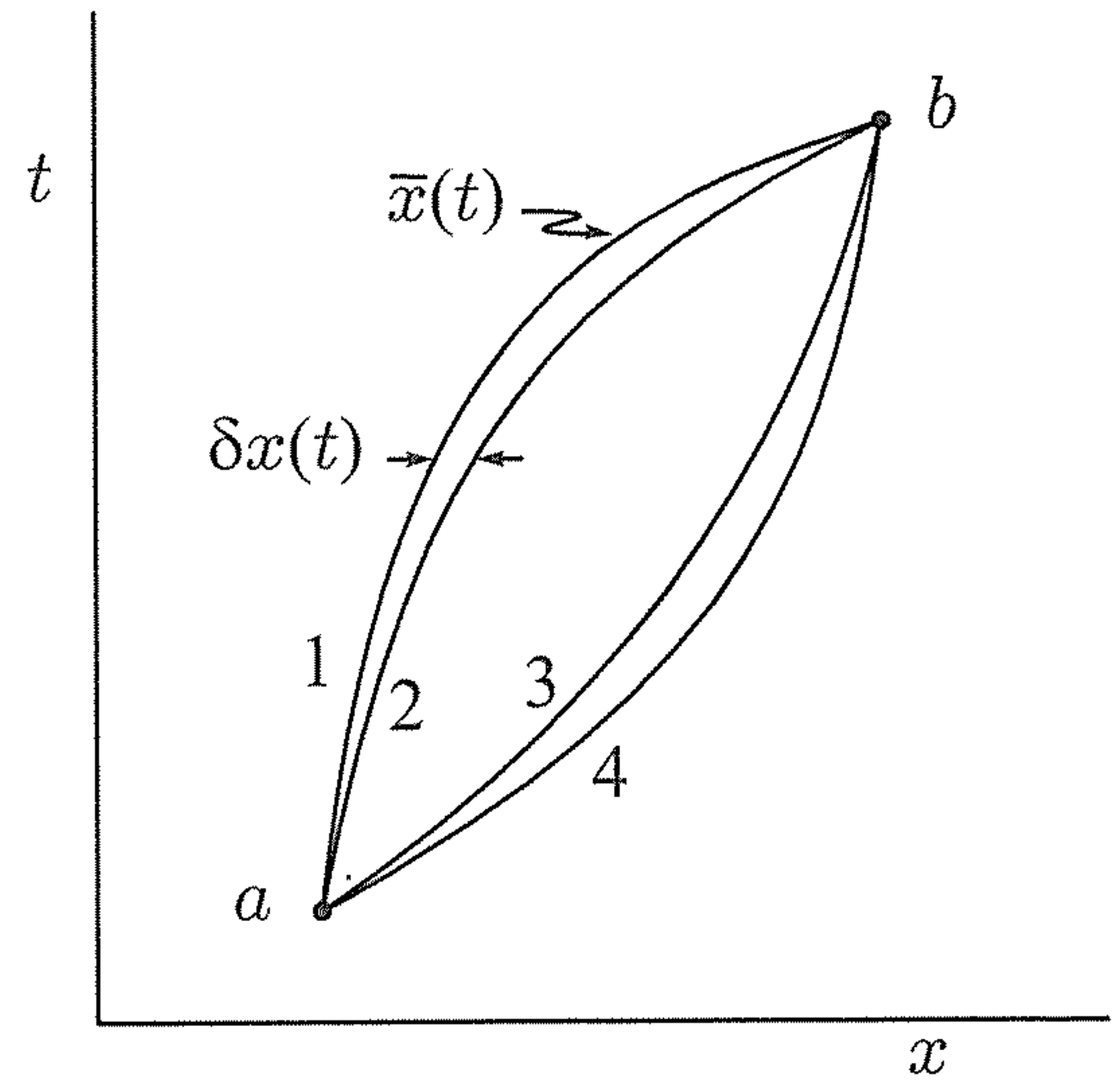
- Far from the classical path, S varies a lot in units of $\hbar$
  - e.g. 3 and 4 contribute with different phases and ***destructively interfere***
- Only paths in the vicinity of $\bar{x}(t)$ have important contributions (that don't cancel), and ***in the classical limit we only need to consider this trajectory***
  - Classical laws of motion arise from the quantum laws!



Defining the path integral

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- How do we construct the path integral/sum?
- $$K(x_B, t_B; x_A, t_A) = (\text{const}) \sum_{\text{all paths}} e^{\frac{i}{\hbar} S_{\text{path}}}$$
- Note the probability amplitude for the particle to travel from position x_A at time t_A to position x_B at time t_B is sometimes called the **kernel** or the **propagator**

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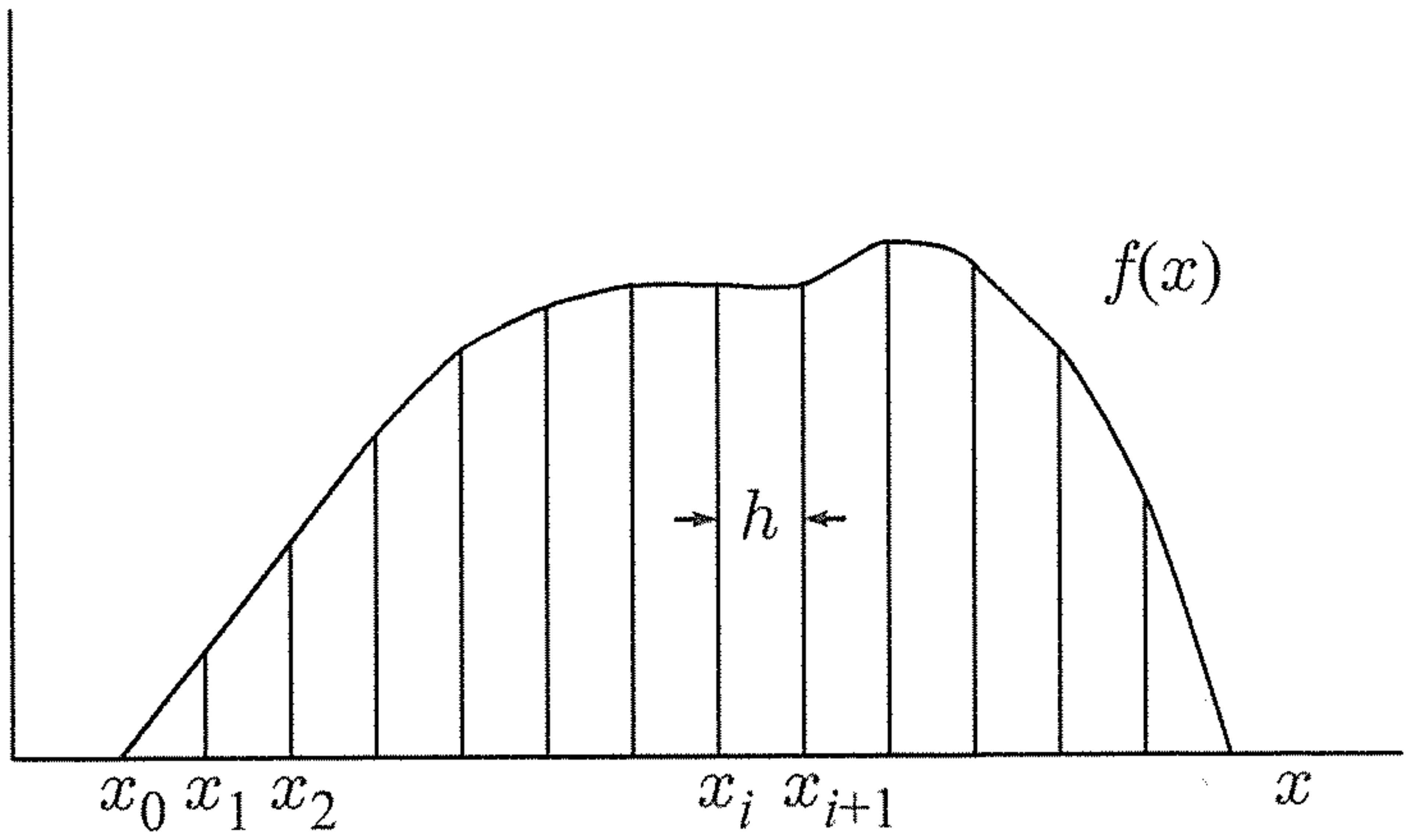
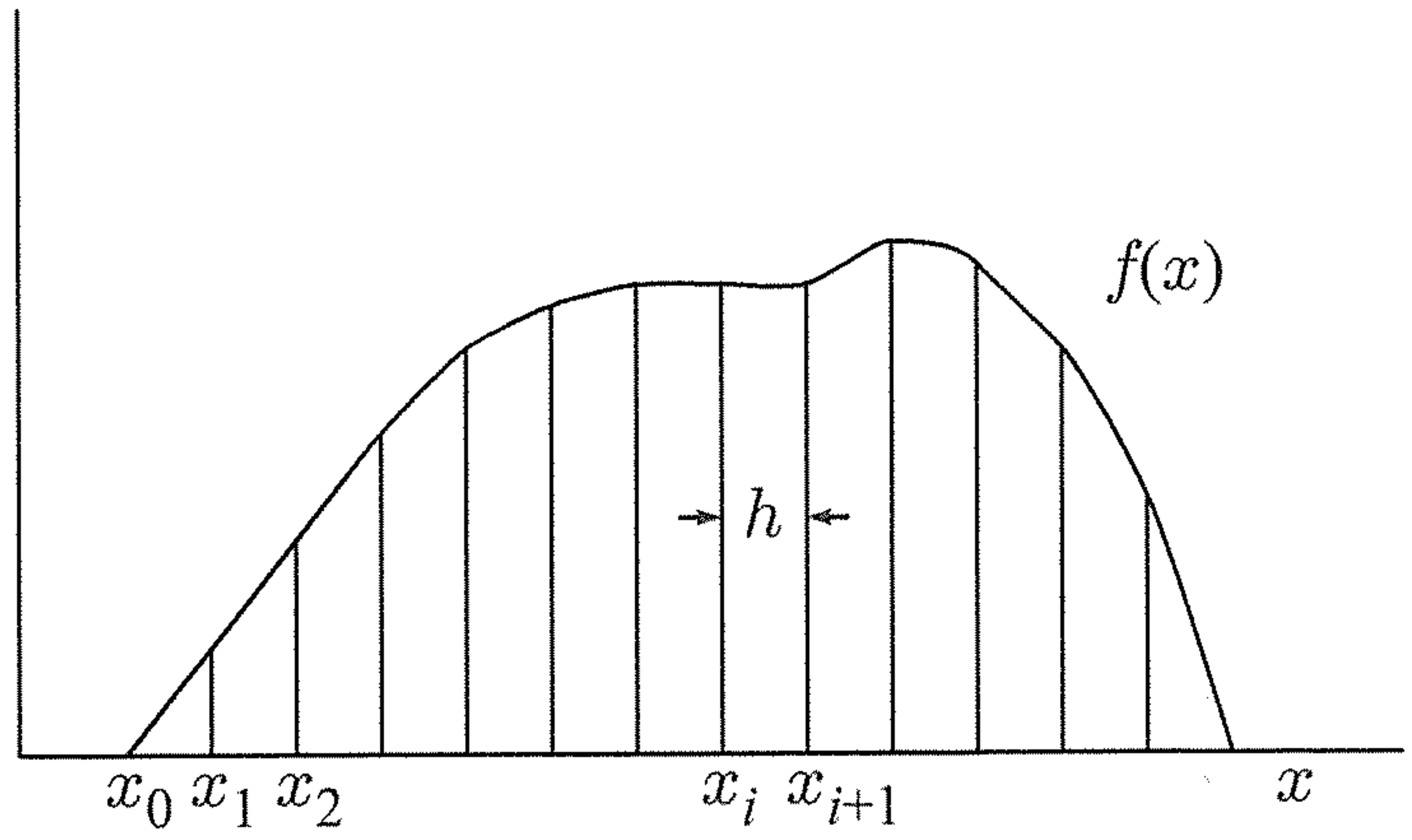
- By analogy with the Riemann integral/sum



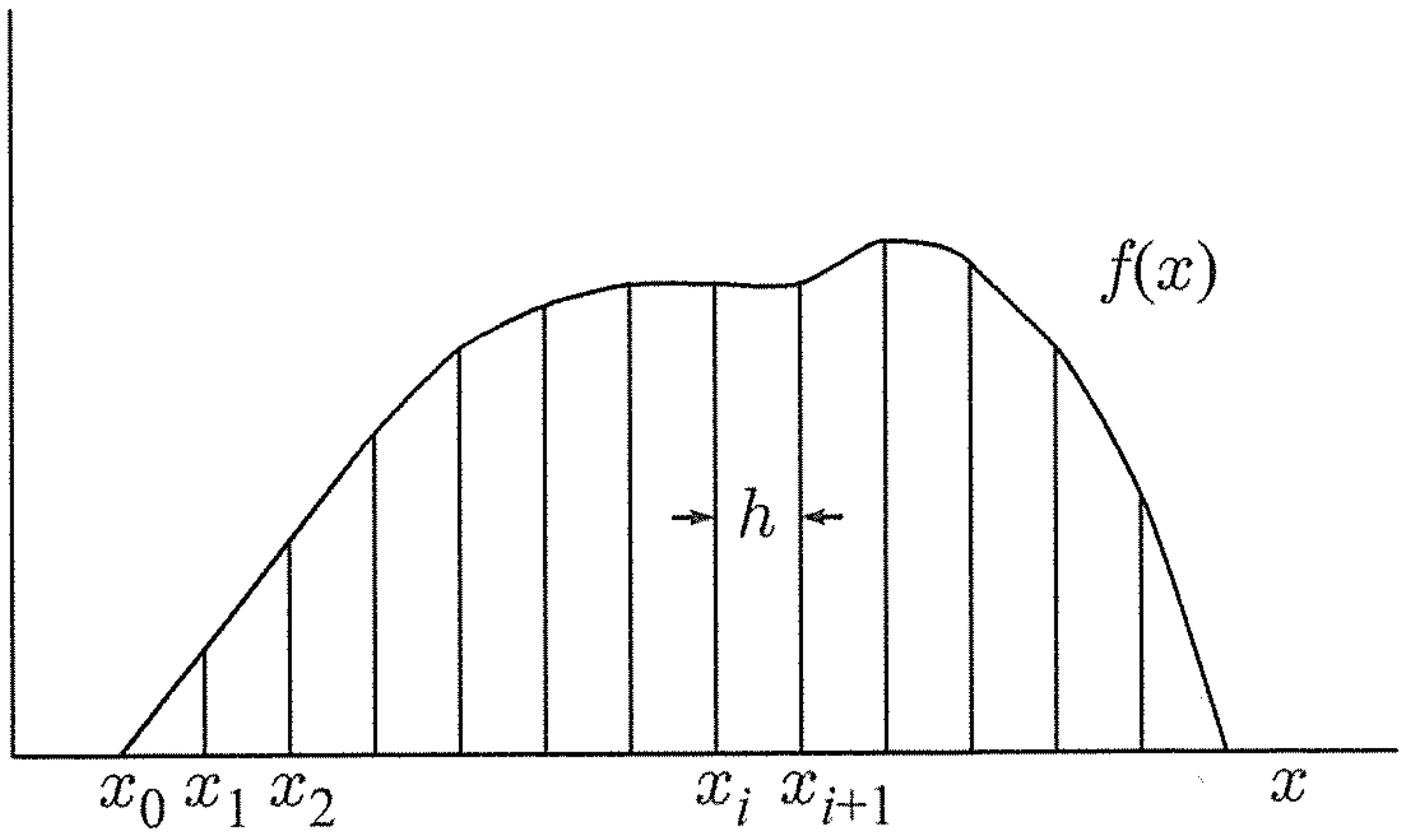
Riemann sum



Riemann sum

- Define N equally spaced x coordinates (separation h) as a representative subset of all coordinates
- Area under the curve A is the sum of all function values evaluated at each x_i, multiplied by an overall normalization factor h,

$$A = \lim_{h\to 0, N\to\infty} h \sum_{i=0}^{N-1} f(x_i)$$



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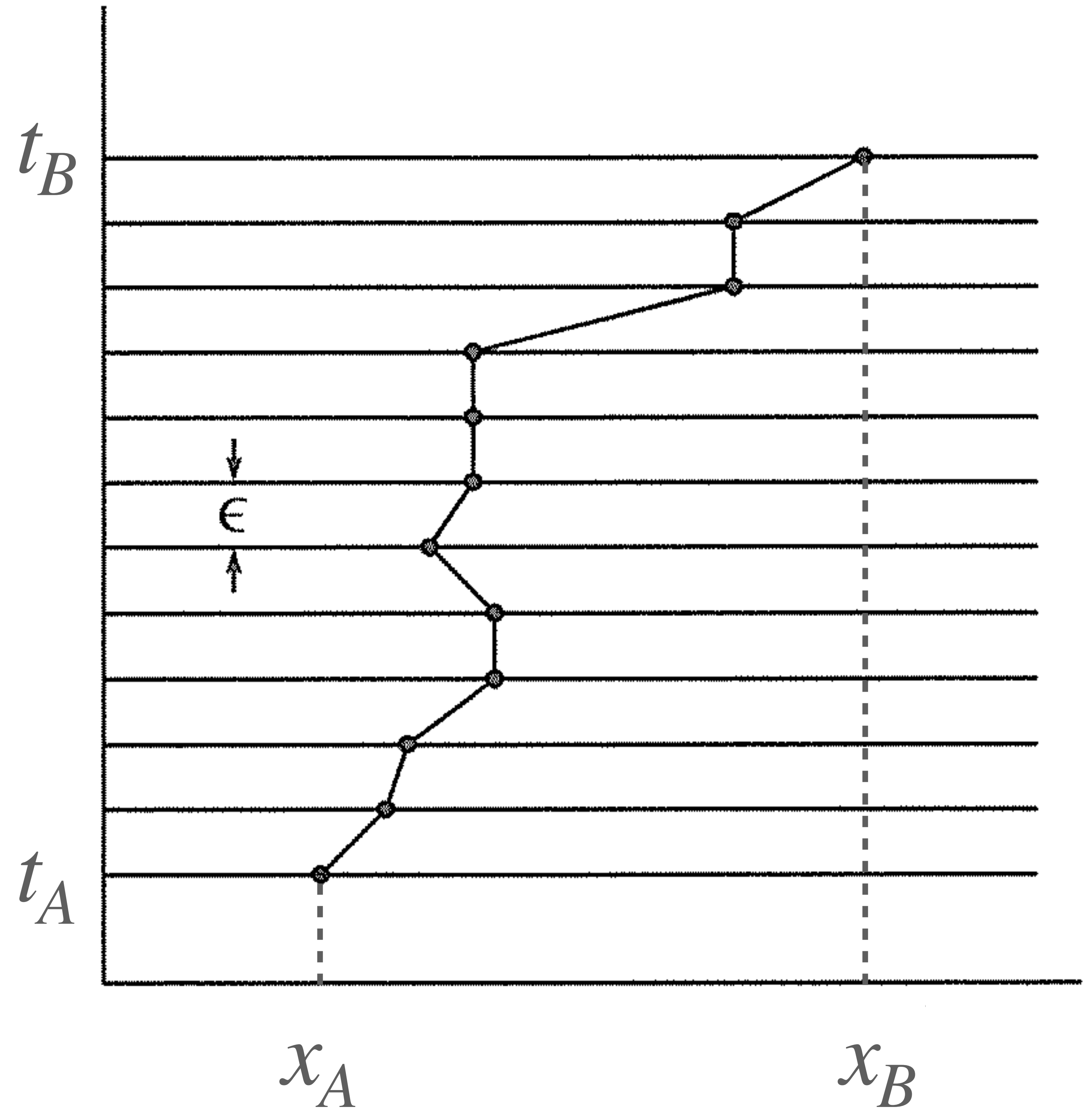

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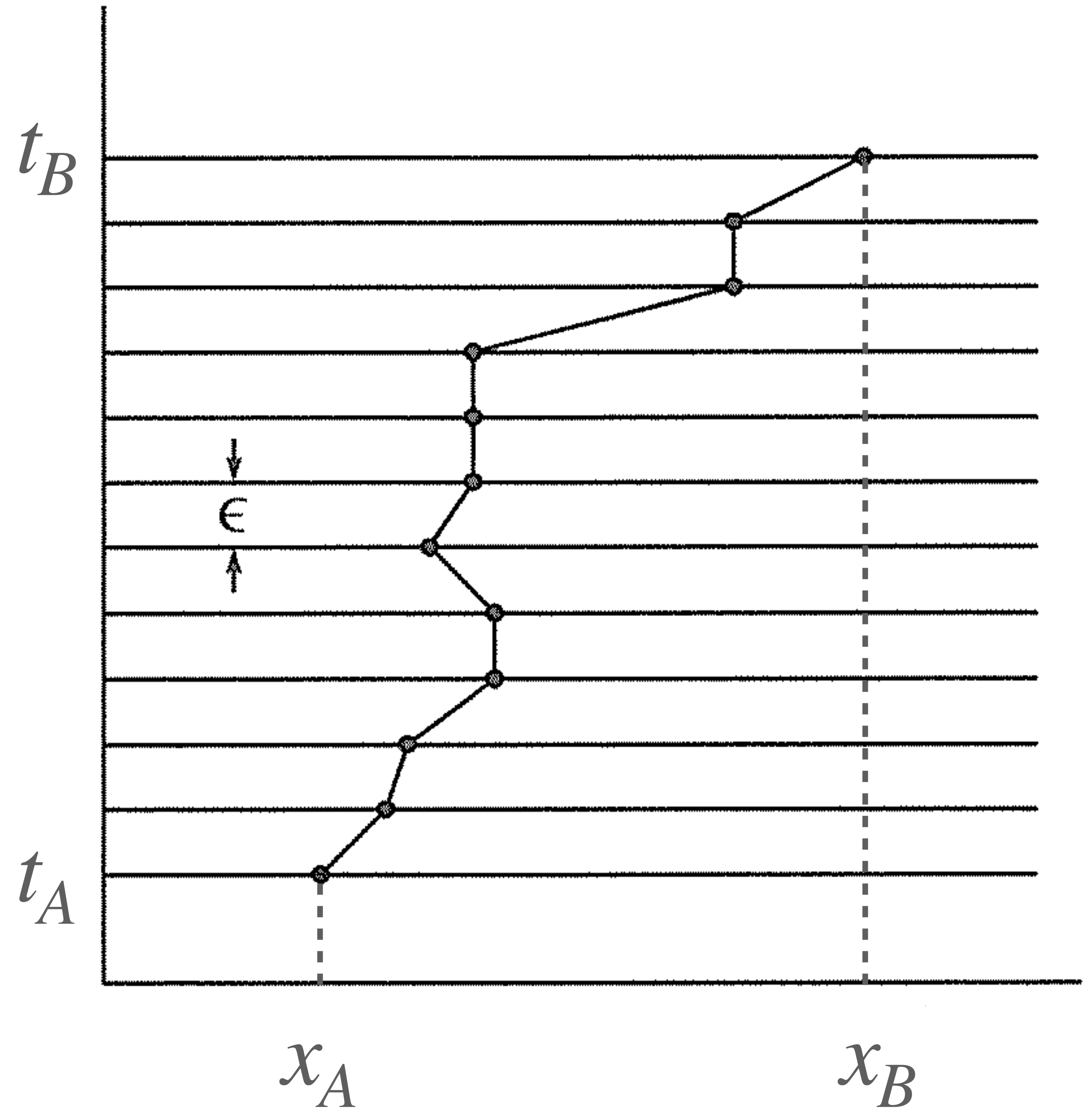

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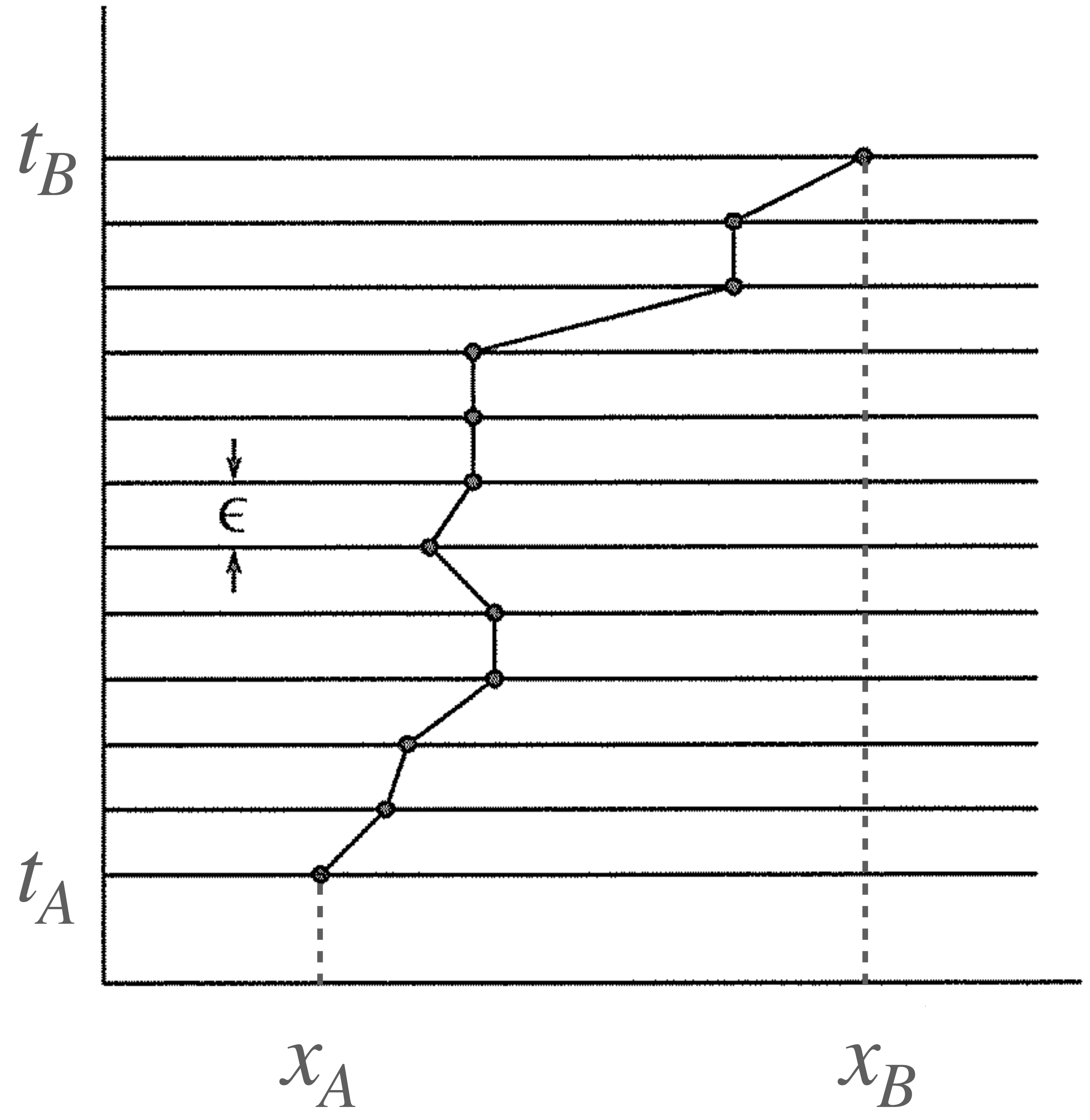

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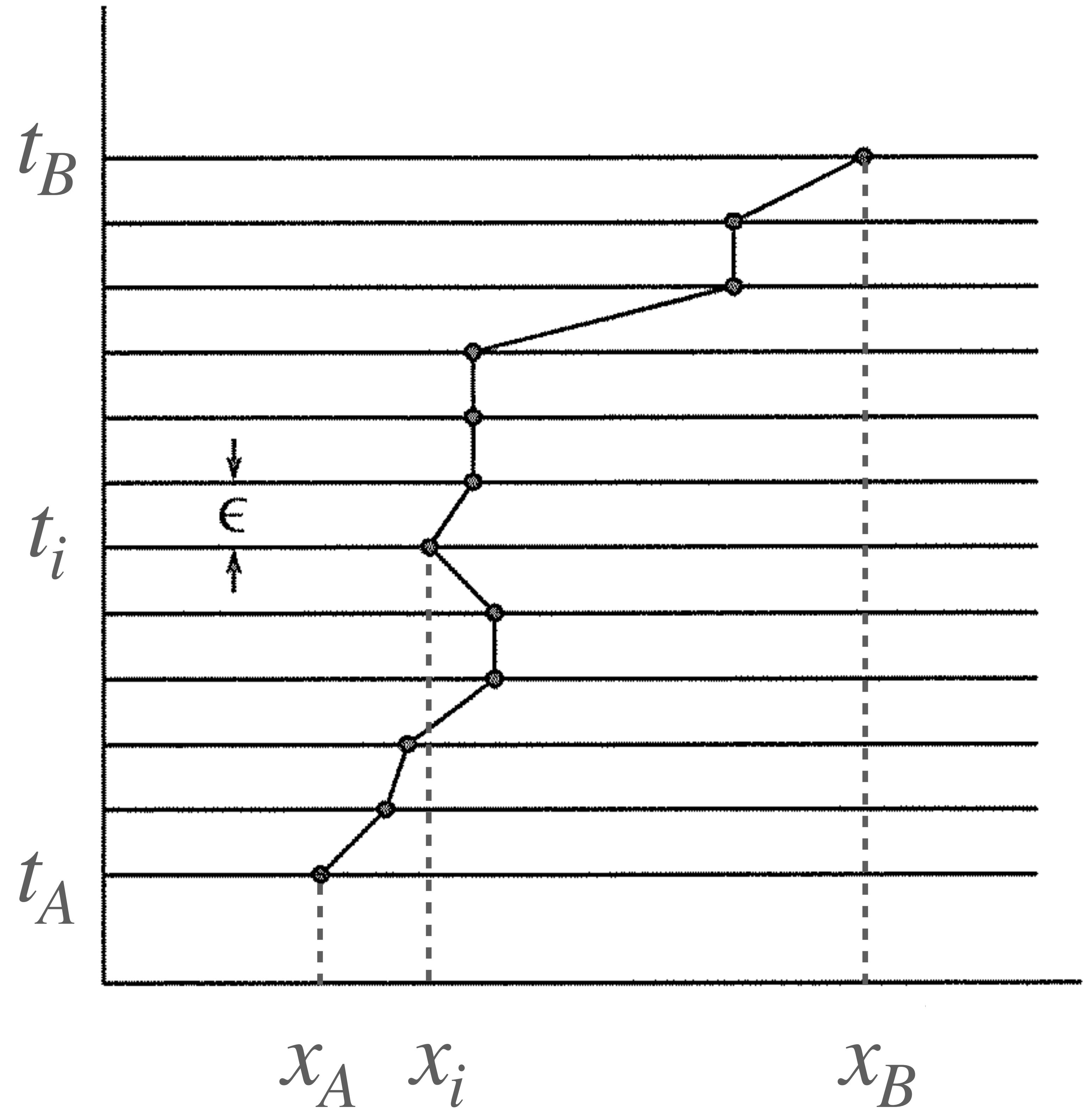

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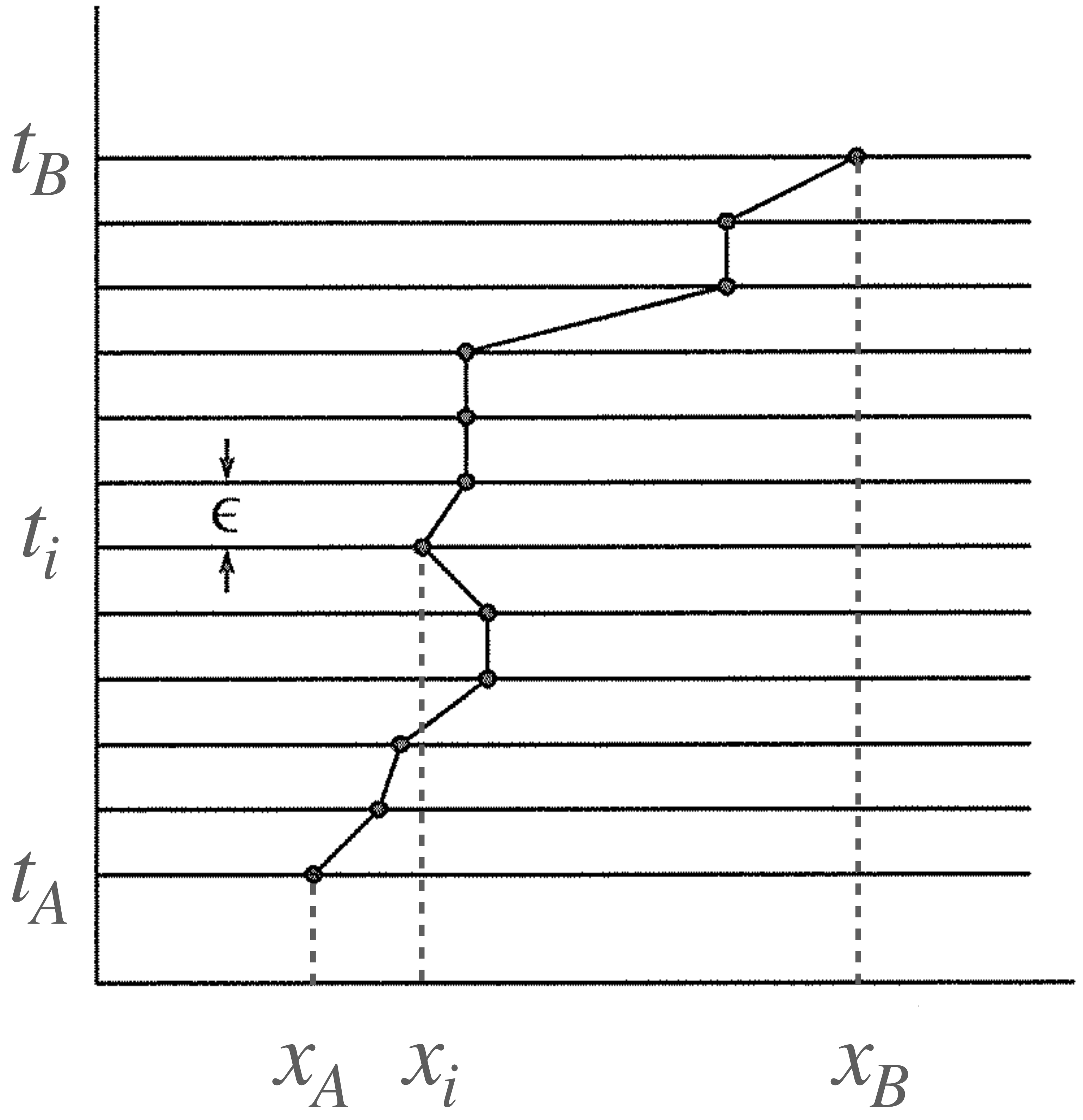
- We construct a path by connecting all the points with straight lines (i.e. constant velocity)



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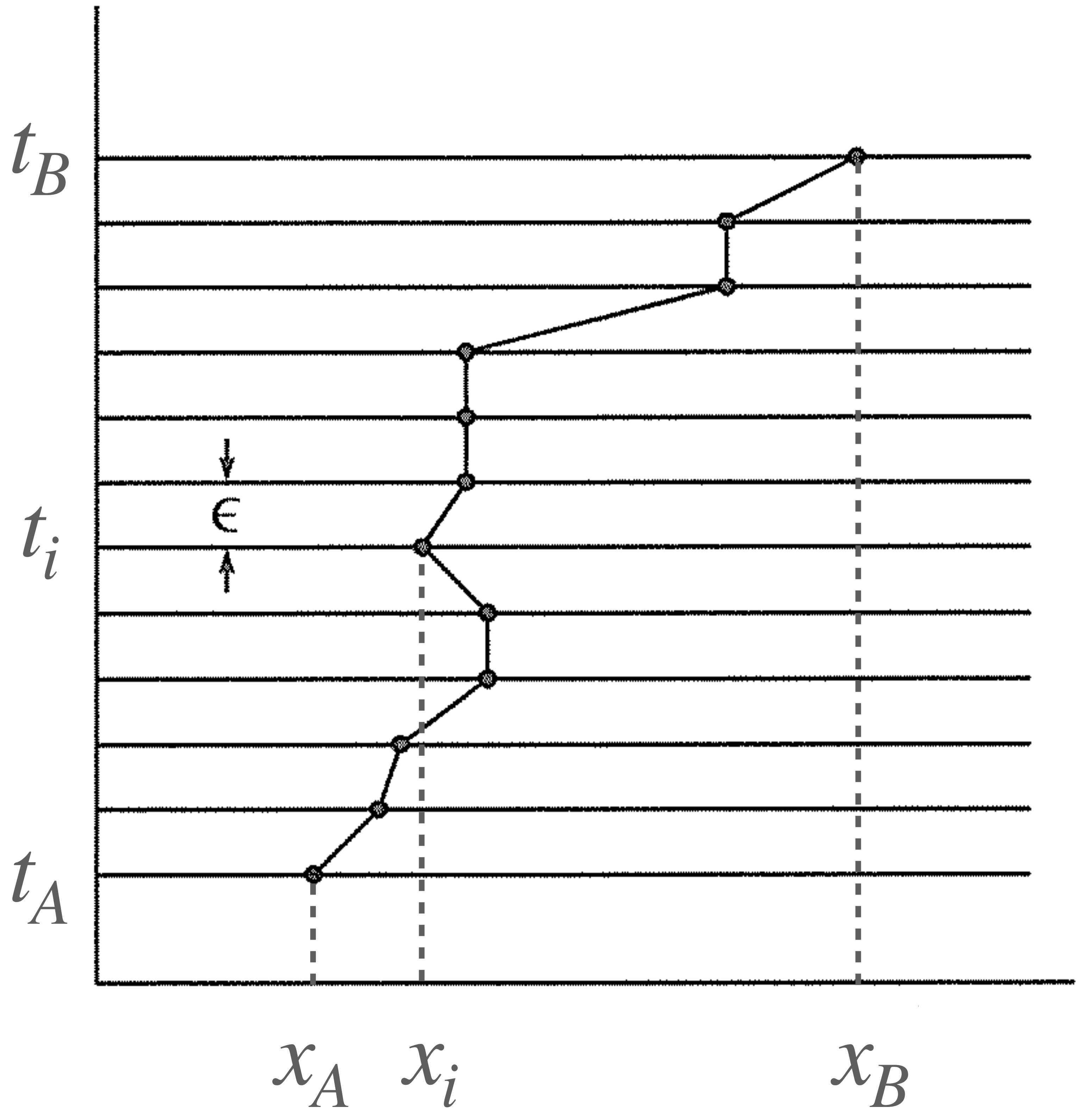

- Summary:

$$N\epsilon = t_B - t_A, \qquad \epsilon = t_{i+1} - t_i, \qquad t_0 = t_A, \qquad t_N = t_B$$
$$x_0 = x_A, \qquad x_N = x_B$$

Defining the path integral

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- Define the sum over all such paths by taking a multiple integral over all x_i coordinate choices
  - $K(x_B, t_B; x_A, t_A) \sim \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} dx_1 dx_2 \cdots dx_{N-1}$
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- For now, we'll take the factor for granted, but we will show how it's derived later
- $K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0,\ N \to \infty} \frac{1}{C} \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} \frac{dx_1}{C} \frac{dx_2}{C} \cdots \frac{dx_{N-1}}{C}$
  - $C = \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{\frac{1}{2}}$; Overall normalizing factor is C^{-N}
  - $S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$

Considerations and notation

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- Note that for the given the path we've defined, the velocities are discontinuous, and therefore the acceleration $\ddot{x}_i$ is formally infinite at each time step t_i!
  - A workaround is to define the acceleration as $\ddot{x}_i = \dfrac{1}{\epsilon}(x_{i+1} - 2x_i + x_{i-1})$, i.e. average over three neighboring points

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- Finally, we can use a shorthand to denote the path integral
  - $K[B, A] = \displaystyle\int_A^B \mathscr{D}x(t) e^{\frac{i}{\hbar}S[B,A]}$

Path integral for a free particle

Path integral for a free particle

- Path integral is $K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar} S[B,A]}$
- So we need to calculate the action for each discretized path from

  $$S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$$

- Note: we can get the action for the full path by adding up the contribution from each component of the path, i.e. assuming $t_A < t_C < t_B$, then the action along any path between A and B is
  - $S[B, A] = S[B, C] + S[C, A]$

Path integral for a free particle

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Putting this together with the result of Quiz 1,

$$S[i, i-1] = \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}, \text{ and}$$

$$S[B, A] = \sum_{i=1}^{N} S[i, i-1] = \sum_{i=1}^{N} \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}$$

So,

$$K(B, A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_N \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{-\frac{N}{2}} \exp\left[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2\right]$$

Note this is a ***Gaussian integral***

Gaussian integrals

Gaussian integrals

Gaussian integral: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

More generally, $\int_{-\infty}^{+\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}$

With complex arguments, $\int_{-\infty}^{+\infty} e^{iax^2+ibx} = \sqrt{\frac{i\pi}{a}} e^{-ib^2/(4a)}$

Gaussian integral proof

Gaussian integral proof

$$\left[\int_{-\infty}^{+\infty} e^{-x^2} dx\right]^2 = \left[\int_{-\infty}^{+\infty} e^{-x^2} dx\right]\left[\int_{-\infty}^{+\infty} e^{-y^2} dy\right]$$

$$= \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2} rdr \qquad (s = r^2)$$

$$= \pi \int_0^{\infty} e^{-s} ds = -\pi e^{-s}\Big|_0^{\infty} = -\pi(0-1) = \pi$$

Back to the path integral

Back to the path integral

We have terms like this we need to integrate:

$$\left(\frac{m}{2\pi i\hbar\epsilon}\right)\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}[(x_2-x_1)^2+(x_1-x_0)^2]\right]dx_1$$

Let's apply our Gaussian integral formula $\displaystyle\int_{-\infty}^{+\infty}e^{iax^2+ibx}=\sqrt{\frac{i\pi}{a}}e^{-ib^2/(4a)}$

$$\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}(x_2^2+2x_1^2+x_0^2-2x_0x_1-2x_1x_2)\right]dx_1=\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}(x_2^2+2x_1^2+x_0^2-2x_0x_1-2x_1x_2)\right]dx_1$$