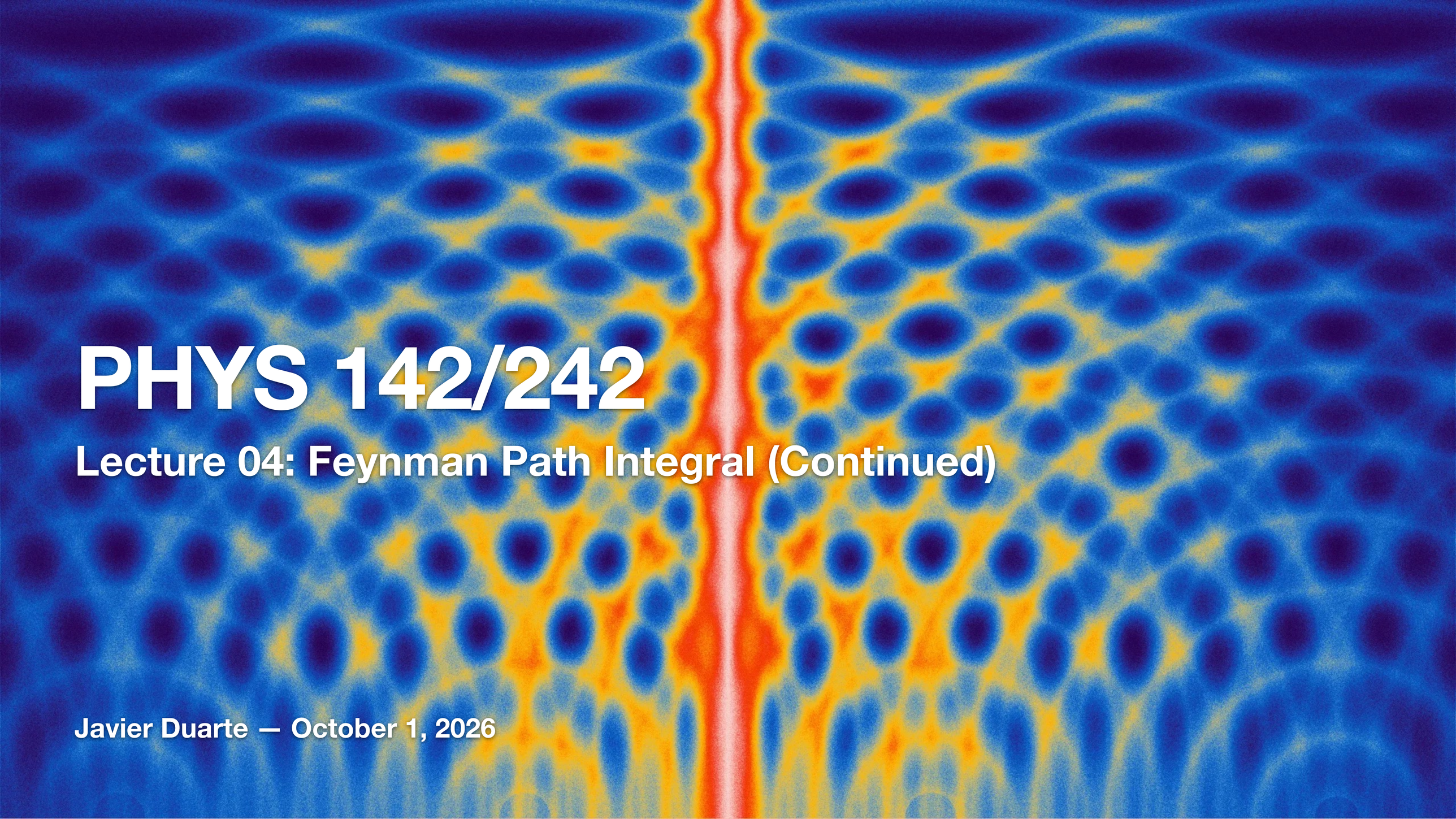


PHYS 142/242

Lecture 04: Feynman Path Integral (Continued)

Javier Duarte — October 1, 2026

Feynman path integral

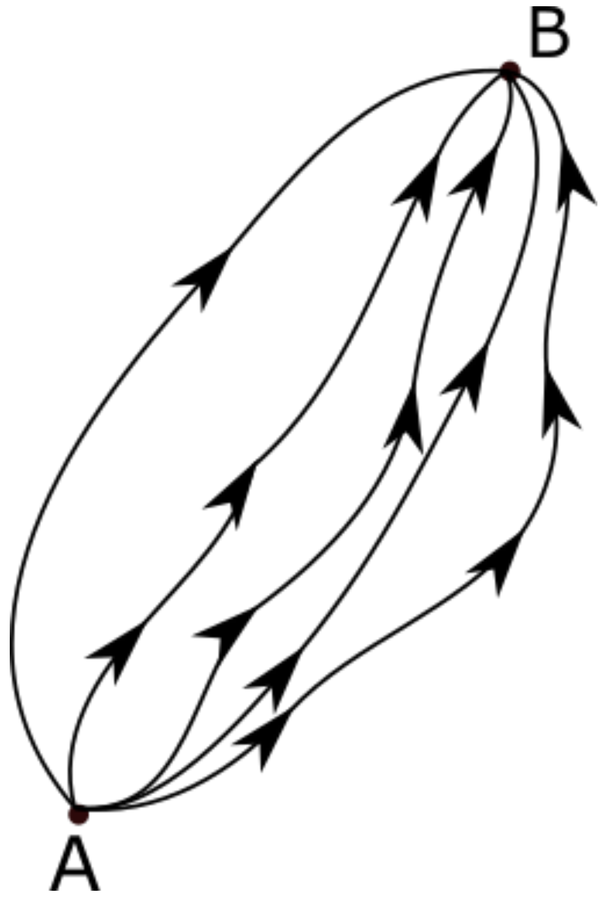
- In quantum mechanics, the amplitude to go from A to B is the sum of contributions $\phi[(x(t)]$ from each path

$$K(B,A) = \sum_{\text{paths from A to B}} \phi[x(t)]$$

- The contribution of a path has a phase proportional to the *classical action* S

$$\phi[x(t)] = (\text{const})e^{(i/\hbar)S[x(t)]}$$

- Generalization of the classical PLA: ***quantum action principle***; reproduces PLA as $\hbar \to 0$ (or equivalently $S \gg \hbar$)



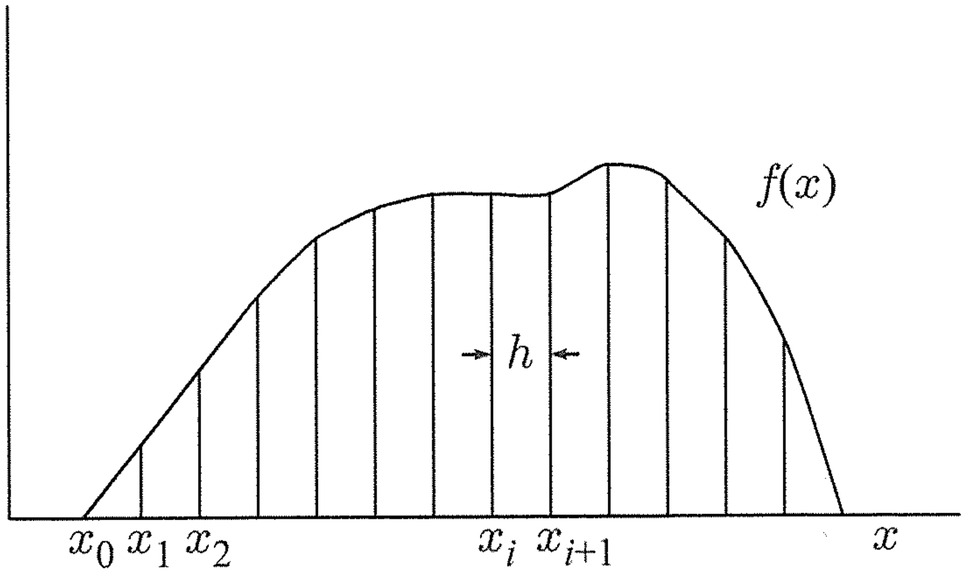
Defining the path integral

- How do we construct the path integral/sum?

$$K(x_B, t_B; x_A, t_A) = (\text{const}) \sum_{\text{all paths}} e^{\frac{i}{\hbar} S_{\text{path}}}$$

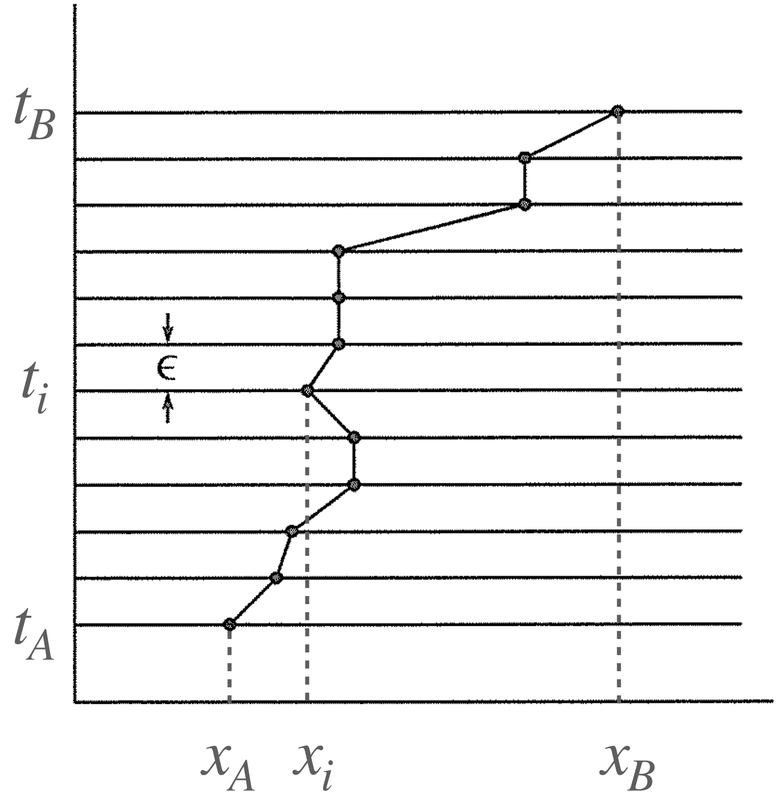
  - Note the probability amplitude for the particle to travel from position x_A at time t_A to position x_B at time t_B is called the **kernel** or the **propagator**

- By analogy with the Riemann integral/sum



Defining the path integral

- First, choose a subset of all paths from (x_A, t_A) to (x_B, t_B) as follows
- Divide time into steps of width ϵ
- At each time t_i, the path passes through some chosen point x_i
- We construct a path by connecting all the points with straight lines (i.e. constant velocity)



- Summary:

$$N\epsilon = t_B - t_A, \qquad \epsilon = t_{i+1} - t_i, \qquad t_0 = t_A, \qquad t_N = t_B$$
$$x_0 = x_A, \qquad x_N = x_B$$

Defining the path integral

- Define the sum over all such paths by taking multiple integrals over all coordinates x_i

$$K(x_B, t_B; x_A, t_A) \sim \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} dx_1 dx_2 \cdots dx_{N-1},$$

where $S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$

- Note: to get the limit to exist we need to know the normalization factor

Defining the path integral

- For now, we'll take the factor for granted, but we will show how it's derived later

$$K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0,\ N \to \infty} \frac{1}{C} \int \cdots \int \int e^{\frac{i}{\hbar} S[B,A]} \frac{dx_1}{C} \frac{dx_2}{C} \cdots \frac{dx_{N-1}}{C},$$

where $C = \left(\frac{2\pi\hbar\epsilon}{m} \right)^{\frac{1}{2}}$

Overall normalizing factor is C^{-N}

Considerations and notation

- For the given the path we've defined, the velocities are discontinuous, and therefore the acceleration $\ddot{x}_i$ is formally infinite at each time step t_i!

- Workaround: define the acceleration as

$$\ddot{x}_i = \frac{1}{\epsilon^2}(x_{i+1} - 2x_i + x_{i-1}),$$

  i.e., average over three neighboring points

- We can use a shorthand to denote the path integral

$$K(B, A) = \int_A^B \mathscr{D}x(t)\ e^{\frac{i}{\hbar}S[B,A]}$$

Path integral for a free particle

- Path integral is

$$K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar} S[B,A]}$$

- So we need to calculate the action for each discretized path from

$$S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$$

- ***Composition***: Action for the full path is the sum of the contributions from each component of the path, i.e., for an intermediate point C along the path,
  - $S[B, A] = S[B, C] + S[C, A]$

Path integral for a free particle

Putting this together with the free particle action

$$S[i, i-1] = \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}, \text{ and}$$

$$S[B, A] = \sum_{i=1}^{N} S[i, i-1] = \sum_{i=1}^{N} \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}$$

So,

$$K(B, A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_N \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{N}{2}} \exp\left[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2\right]$$

Note this is a ***Gaussian integral***

Gaussian integrals

Gaussian integral: $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$

More generally, $\int_{-\infty}^{+\infty} e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)}$

With complex arguments, $\int_{-\infty}^{+\infty} e^{iax^2+ibx} = \sqrt{\frac{i\pi}{a}} e^{-ib^2/(4a)}$

Gaussian integral proof

$$\left[\int_{-\infty}^{+\infty} e^{-x^2} dx\right]^2 = \left[\int_{-\infty}^{+\infty} e^{-x^2} dx\right]\left[\int_{-\infty}^{+\infty} e^{-y^2} dy\right]$$

$$= \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dxdy$$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} rdrd\theta$$

$$= 2\pi\int_0^{\infty} e^{-r^2} rdr \qquad (s = r^2)$$

$$= \pi\int_0^{\infty} e^{-s} ds = -\pi e^{-s}\Big|_0^{\infty} = -\pi(0-1) = \pi$$

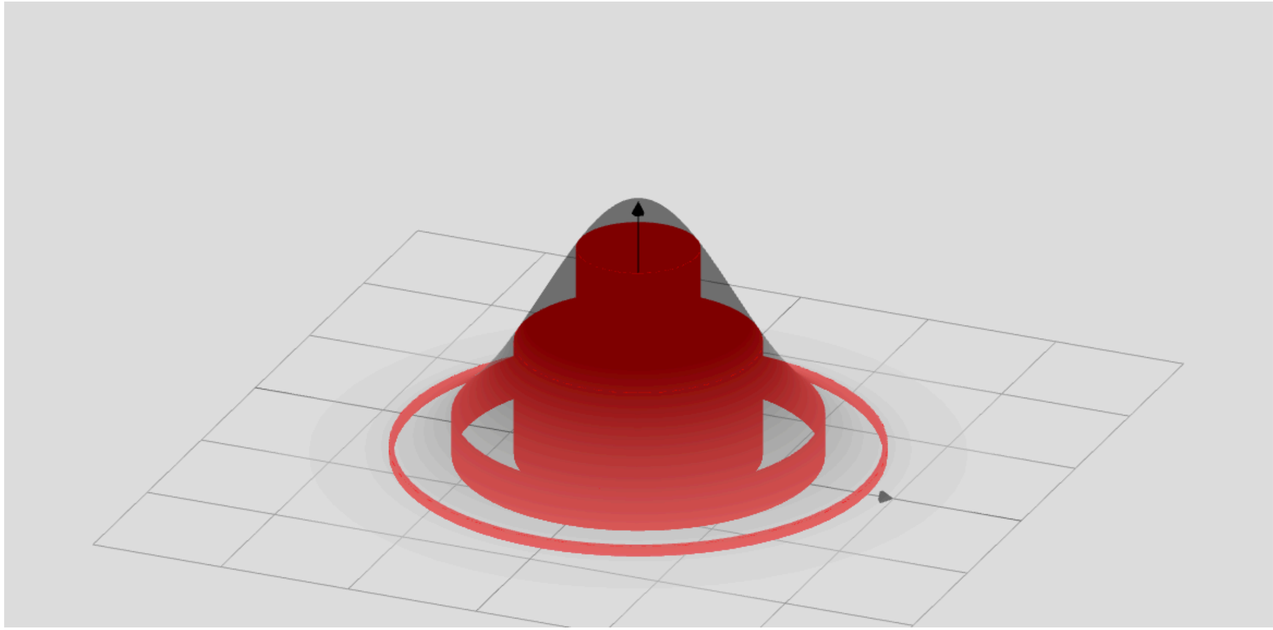
Gaussian integral proof by visualization

https://idan-alter.github.io/2023/02/20/Gaussian-Integral.html

$$\text{Area of cylinder} = 2\pi r e^{-r^2}$$

$$\int_0^\infty 2\pi r e^{-r^2} dr$$

$$\int_0^\infty 2\pi r e^{-r^2} dr = 2\pi \frac{-e^{-r^2}}{2}\Bigg|_0^\infty = \pi$$



Back to the path integral

We have terms like this we need to integrate:

$$\left(\frac{m}{2\pi i\hbar\epsilon}\right)\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}[(x_2-x_1)^2+(x_1-x_0)^2]\right]dx_1$$

Let's apply our Gaussian integral formula $\int_{-\infty}^{+\infty} e^{iax^2+ibx+c}=\sqrt{\frac{i\pi}{a}}e^{-ib^2/(4a)+c}$

$$\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}(x_2^2+2x_1^2+x_0^2-2x_0x_1-2x_1x_2)\right]dx_1=\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}(2x_1^2-2(x_0+x_2)x_1+(x_0^2+x_2^2)\right]dx_1$$

So $a=\frac{m}{\hbar\epsilon}, b=-\frac{m}{\hbar\epsilon}(x_0+x_2), c=\frac{im}{2\hbar\epsilon}(x_0^2+x_2^2)$

Back to the path integral

$$\sqrt{\frac{i\pi}{a}}e^{-ib^2/(4a)+c} = \sqrt{\frac{i\pi\hbar\epsilon}{m}}\exp\left[-\frac{im}{4\hbar\epsilon}(x_0+x_2)^2+\frac{im}{2\hbar\epsilon}(x_0^2+x_2^2)\right]$$
$$= \sqrt{\frac{i\pi\hbar\epsilon}{m}}\exp\left[-\frac{im}{4\hbar\epsilon}(x_0^2+2x_0x_2+x_2^2-2x_0^2-2x_2^2)\right]$$
$$= \sqrt{\frac{i\pi\hbar\epsilon}{m}}\exp\left[\frac{im}{4\hbar\epsilon}(x_2-x_0)^2\right]$$

Now, we can figure out the original integral

$$\left(\frac{m}{2\pi i\hbar\epsilon}\right)\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar\epsilon}[(x_2-x_1)^2+(x_1-x_0)^2]\right]dx_1 = \left(\frac{m}{2\pi i\hbar(2\epsilon)}\right)^{1/2}\exp\left[\frac{im}{2\hbar(2\epsilon)}(x_2-x_0)^2\right]$$

Next integral

Now what if we multiply the existing result by the next term and integrate over x_2?

$$\left(\frac{m}{2\pi i\hbar(2\epsilon)}\right)^{1/2}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}\int_{-\infty}^{\infty}\exp\left[\frac{im}{2\hbar(2\epsilon)}(x_2-x_0)^2+\frac{im}{2\hbar\epsilon}(x_3-x_2)^2\right]dx_2$$

$$=\left(\frac{m}{2\pi i\hbar(3\epsilon)}\right)^{1/2}\exp\left[\frac{im}{2\hbar(3\epsilon)}(x_3-x_0)^2\right]$$

Recursive pattern

A pattern emerges… After $N-1$ steps, we get

$$K(B,A) = \lim_{\epsilon\to 0} \int \cdots \int dx_1 \cdots dx_N \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{N}{2}} \exp\left[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2\right]$$

$$= \lim_{\epsilon\to 0} \left(\frac{m}{2\pi i\hbar(N\epsilon)}\right)^{1/2} \exp\left[\frac{im}{2\hbar(N\epsilon)}(x_N - x_0)^2\right]$$

$$= \left(\frac{m}{2\pi i\hbar(t_B - t_A)}\right)^{1/2} \exp\left[\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)}\right]$$

because $N\epsilon = t_B - t_A, x_0 = x_A, x_N = x_B$

Free particle behavior

The free particle propagator encodes a lot of ***physics***

For convenience, let's analyze it when $A = (0,0)$ and $B = (x, t)$; i.e. a particle is traveling from the origin to position x in time t

$$K(x, t; 0,0) = \left(\frac{m}{2\pi i\hbar t} \right)^{1/2} \exp \left[\frac{imx^2}{2\hbar t} \right]$$

At fixed t, let's see what the propagator looks like as a function of position x

Then let's do the same at fixed x, for the propagator as a function of time t

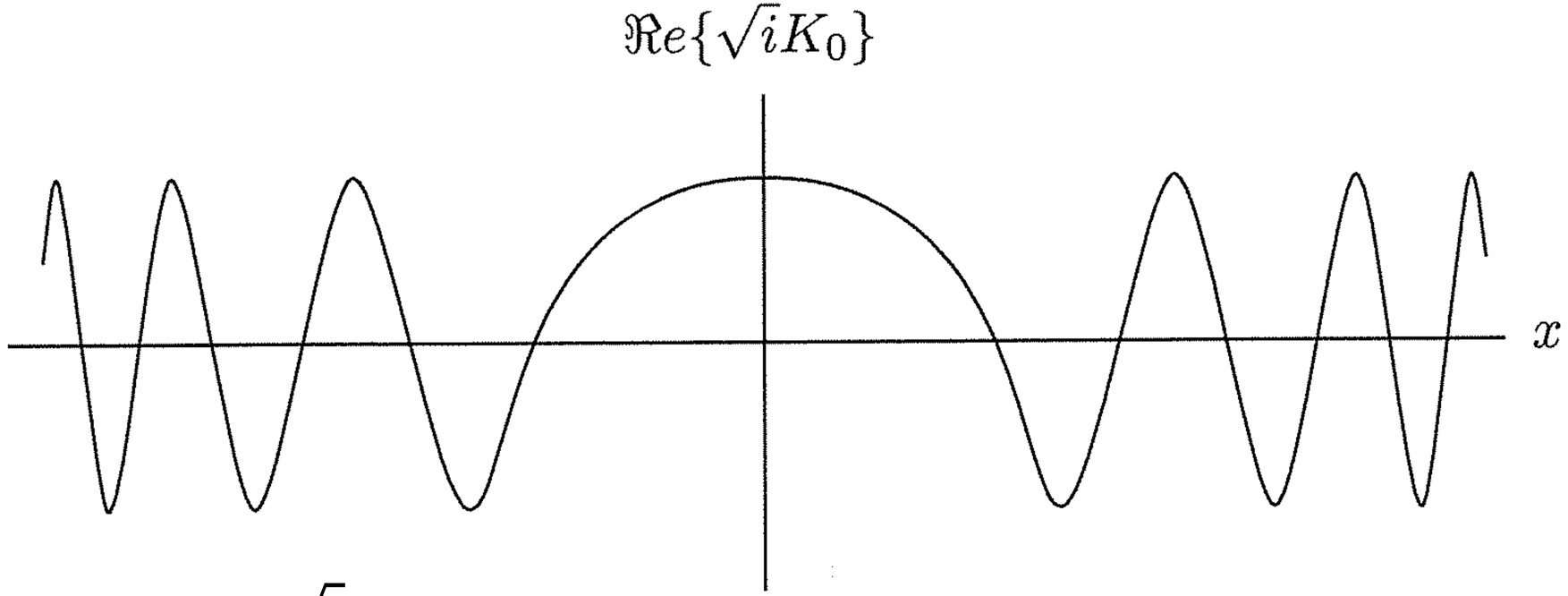
Reminder of plane waves

Plane wave solution to 1D Schrödinger equation

$$\psi(x,t) = C\exp[i(kx - \omega t)]$$

means particle has momentum $p = \hbar k$ and energy $E = \hbar\omega$

Free particle behavior at large x



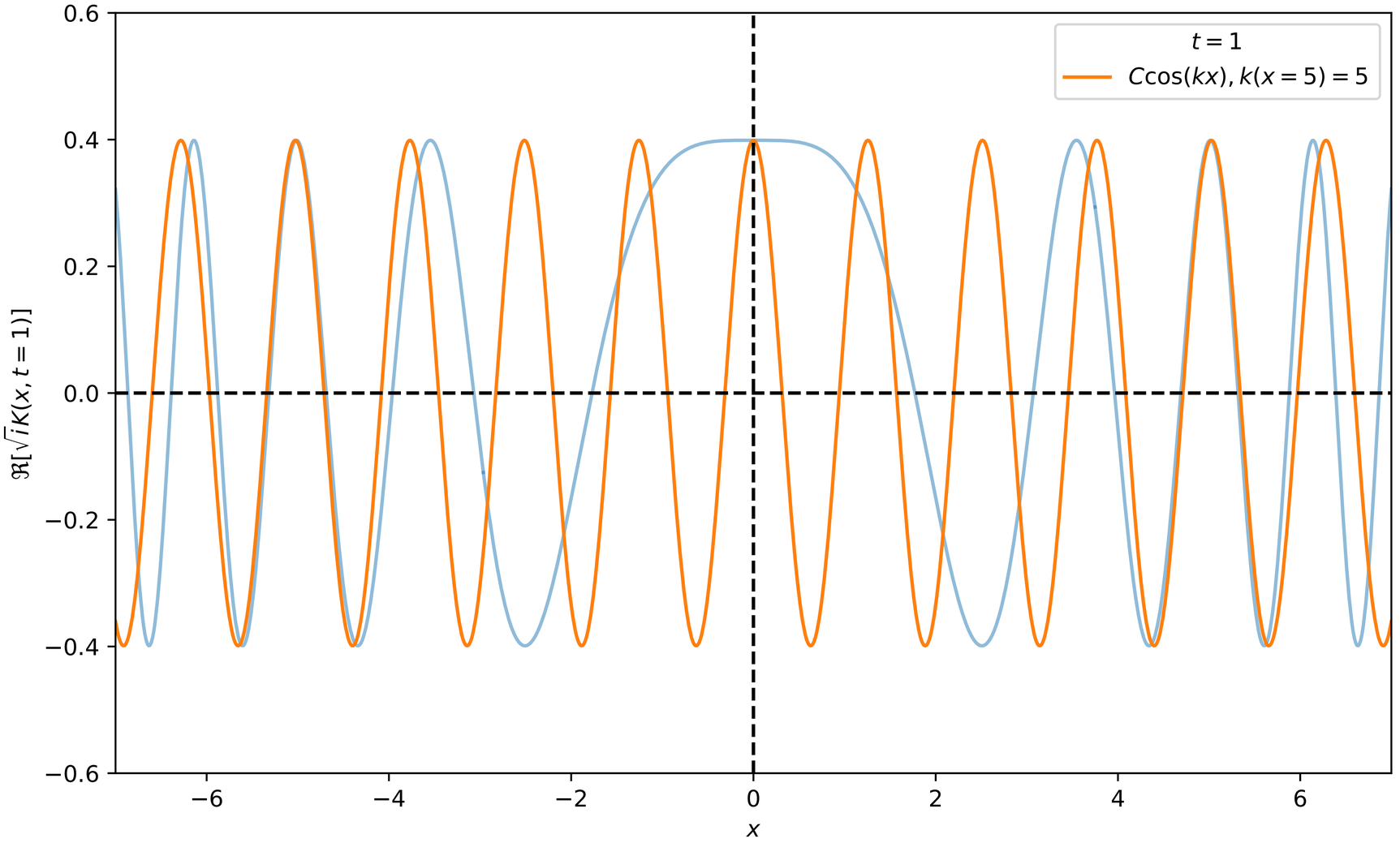
At large x (fixed t), $\Re[\sqrt{i}K]$ oscillates with slowly varying wavelength λ

$$2\pi = \frac{m(x+\lambda)^2}{2\hbar t} - \frac{mx^2}{2\hbar t} = \frac{mx\lambda}{\hbar t} + \frac{m\lambda^2}{2\hbar t} \Rightarrow \lambda \approx \frac{2\pi\hbar}{m(x/t)}$$

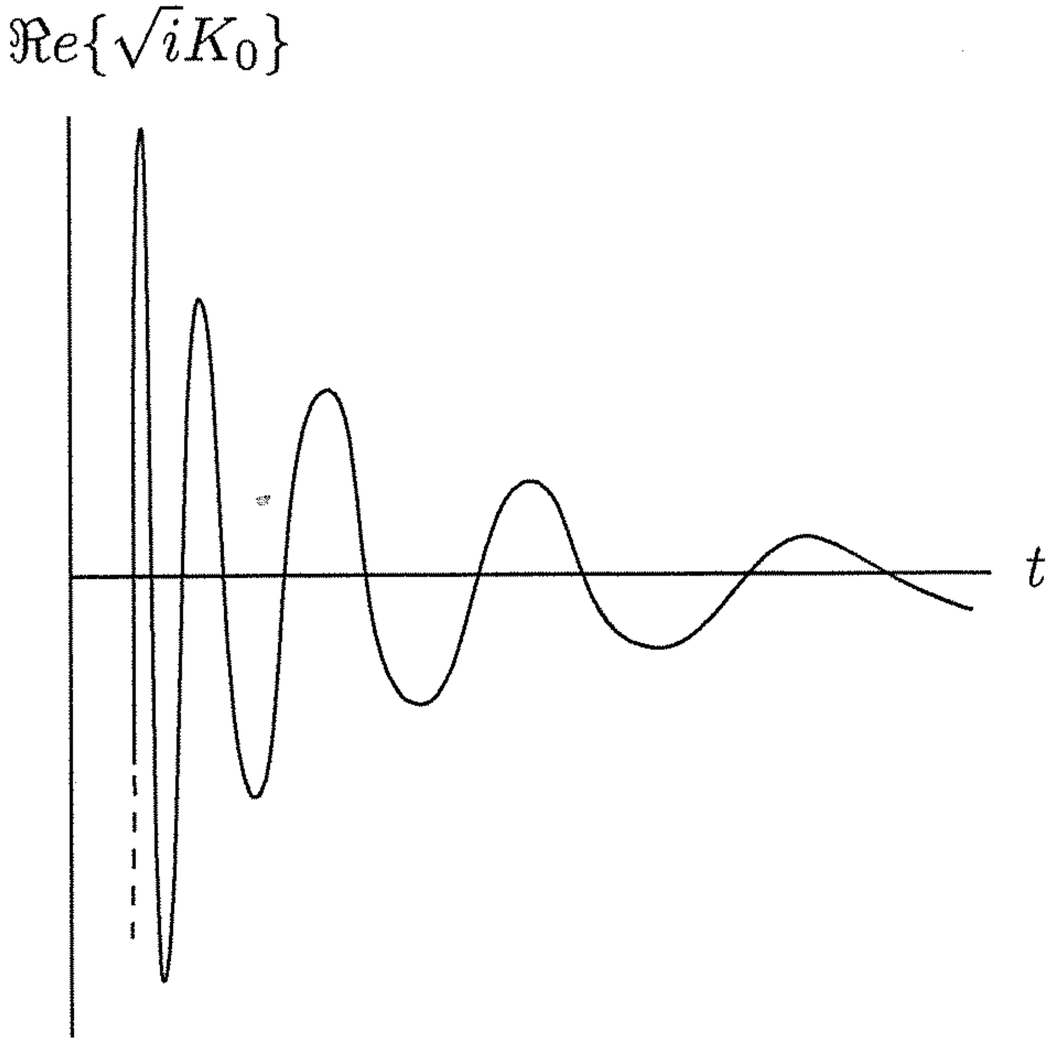
assuming $x \gg \lambda$

$k = \frac{2\pi}{\lambda} = \frac{m(x/t)}{\hbar}$ and $p = \hbar k$!

Compare to plane wave at $(x = 5, t = 1)$



Free particle behavior at large t



At large t (fixed x), $\Re[\sqrt{i}K]$ amplitude and period varies; Neglecting change in amplitude,

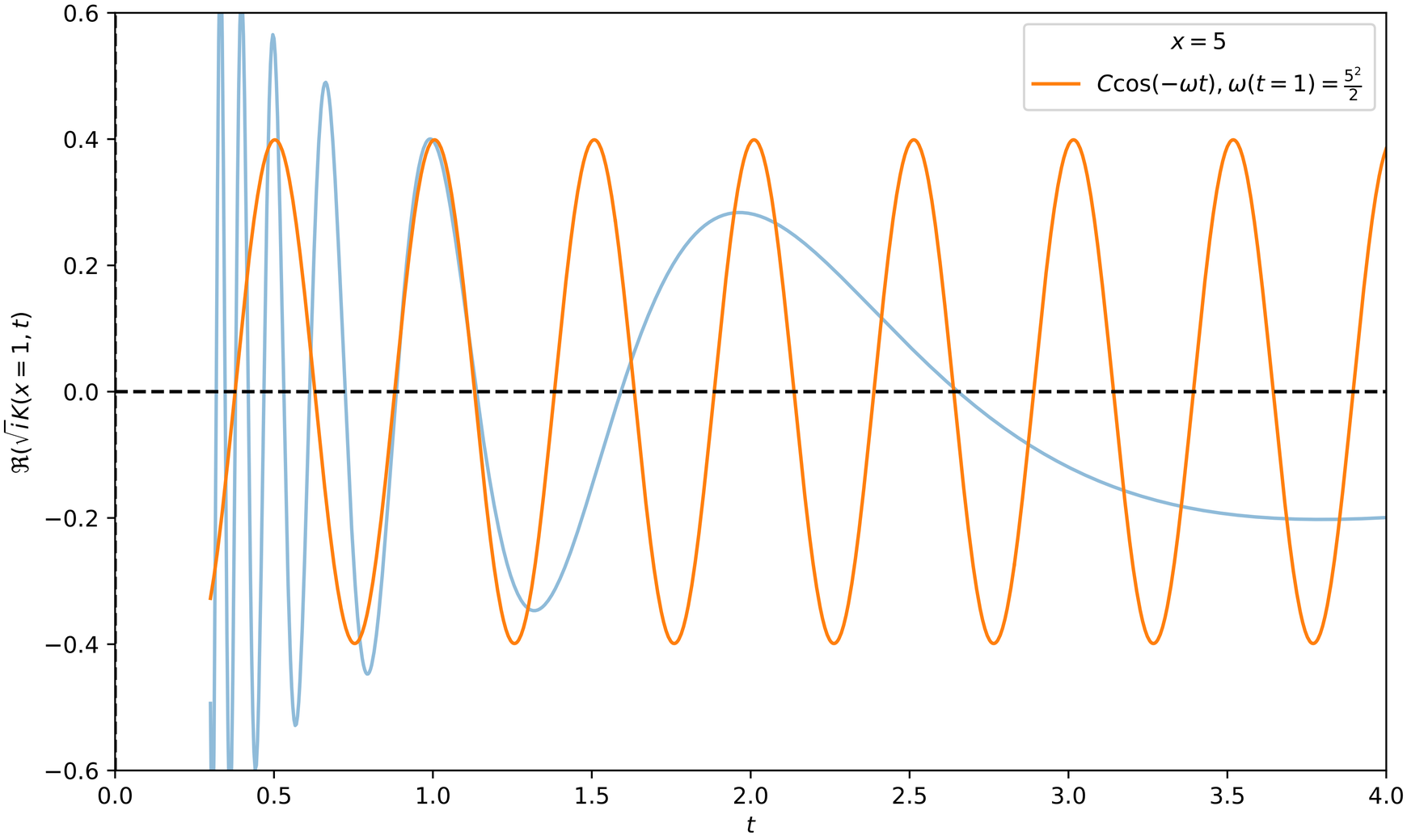
$$2\pi = \frac{mx^2}{2\hbar t} - \frac{mx^2}{2\hbar(t+T)} = \frac{mx^2}{2\hbar t}\left(\frac{T}{1+T/t}\right)$$

$$\approx \frac{mT}{2\hbar}\left(\frac{x}{t}\right)^2 \Rightarrow T \approx \frac{\pi\hbar}{m(x/t)^2}$$

assuming $t \gg T$

$$\omega = \frac{2\pi}{T} = \frac{m}{2\hbar}\left(\frac{x}{t}\right)^2 \text{ and } E = \hbar\omega \text{ !}$$

Compare to plane wave at $(x = 5, t = 1)$



Energy and momentum and waves

Concepts of momentum and energy are extended to quantum mechanics as expected

1. If amplitude varies in space as e^{ikx}, say that the particle has momentum $\hbar k$

2. If amplitude varies in time as $e^{-i\omega t}$, say that the particle has energy $\hbar\omega$