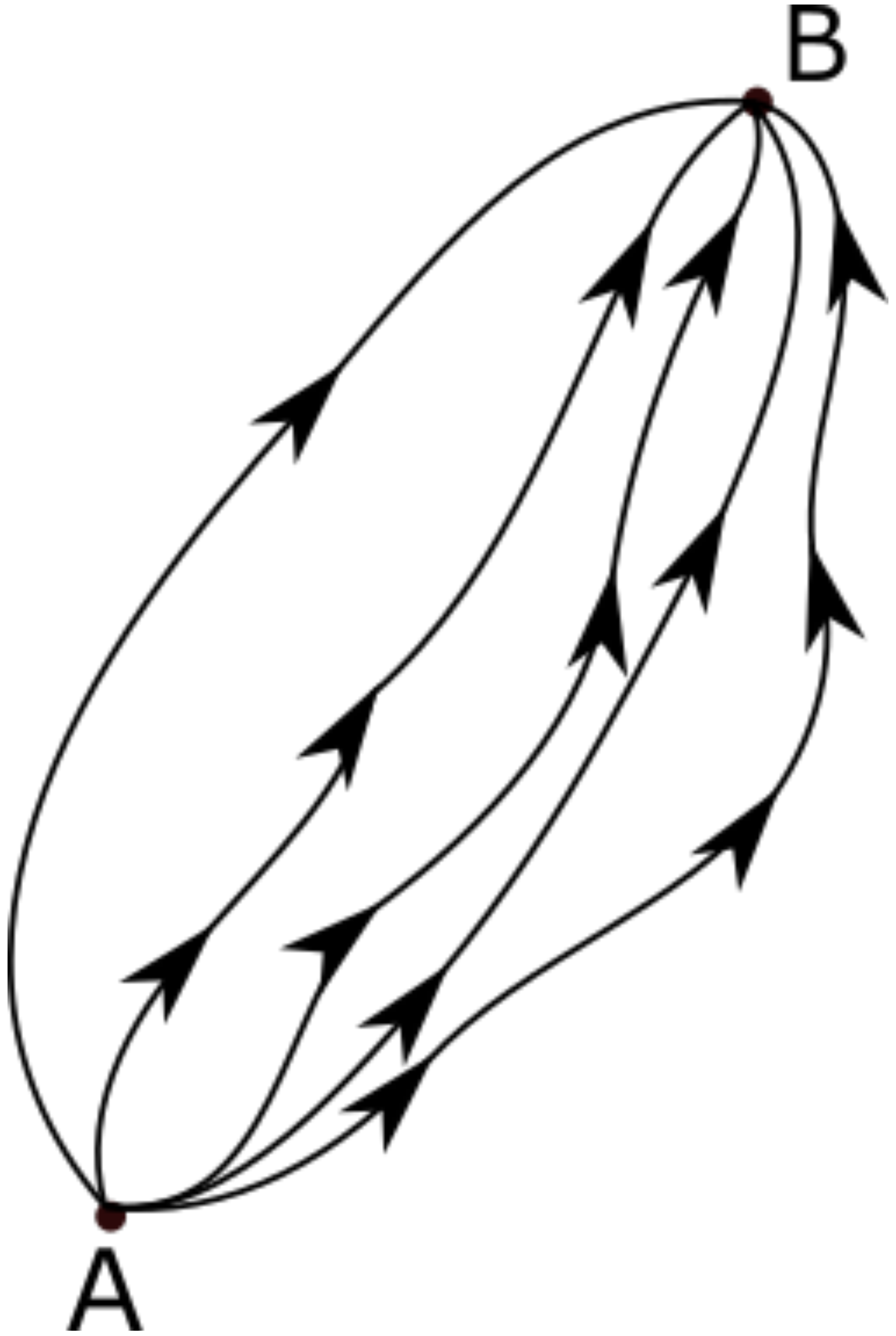
PHYS 142/242

Lecture 04: Free Particle

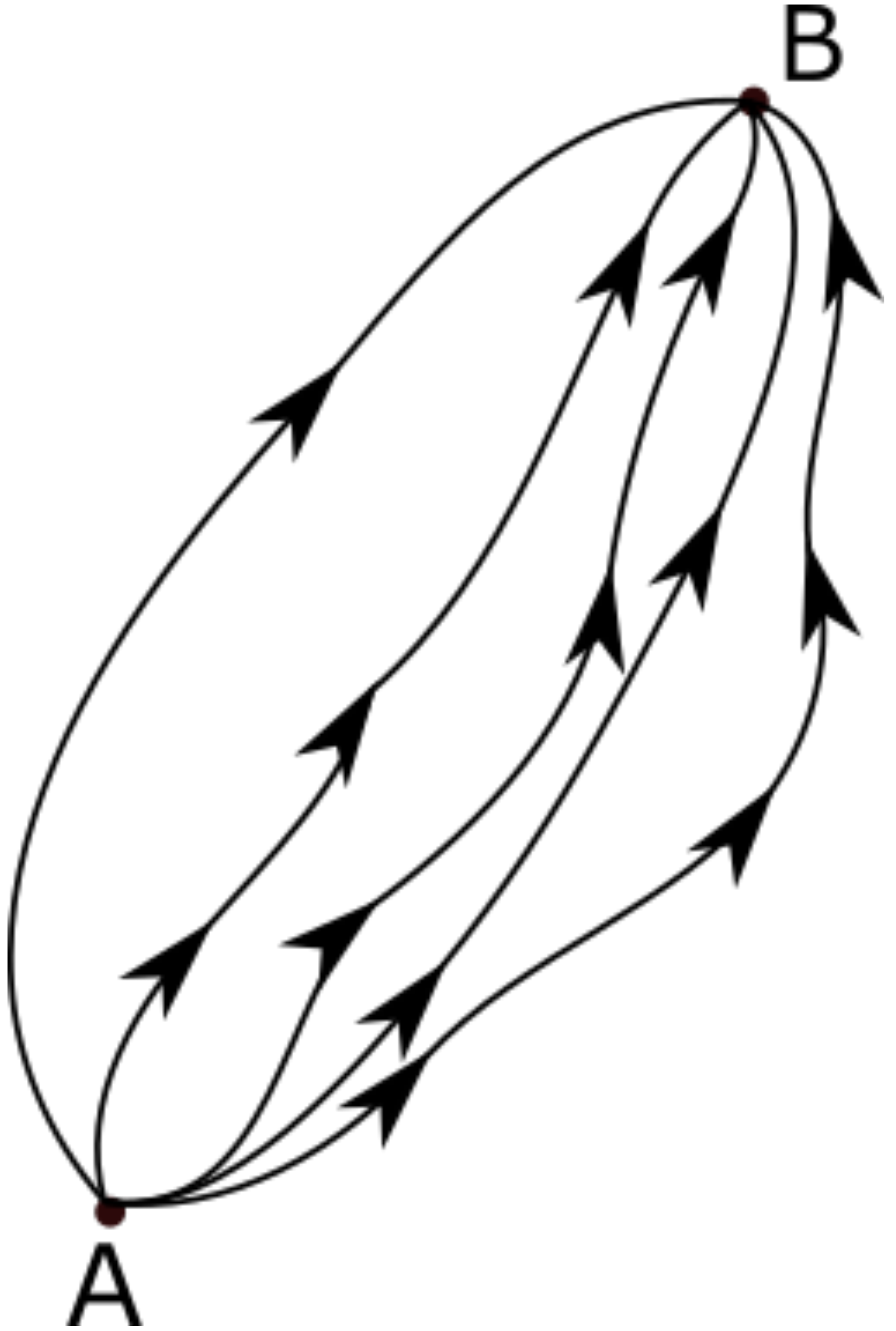
Javier Duarte — January 13, 2025

Feynman path integral



Feynman path integral

- In quantum mechanics, the amplitude to go from A to B is the sum of contributions $\phi[(x(t)]$ from each path
- $$K(B,A) = \sum_{\text{paths from A to B}} \phi[x(t)]$$



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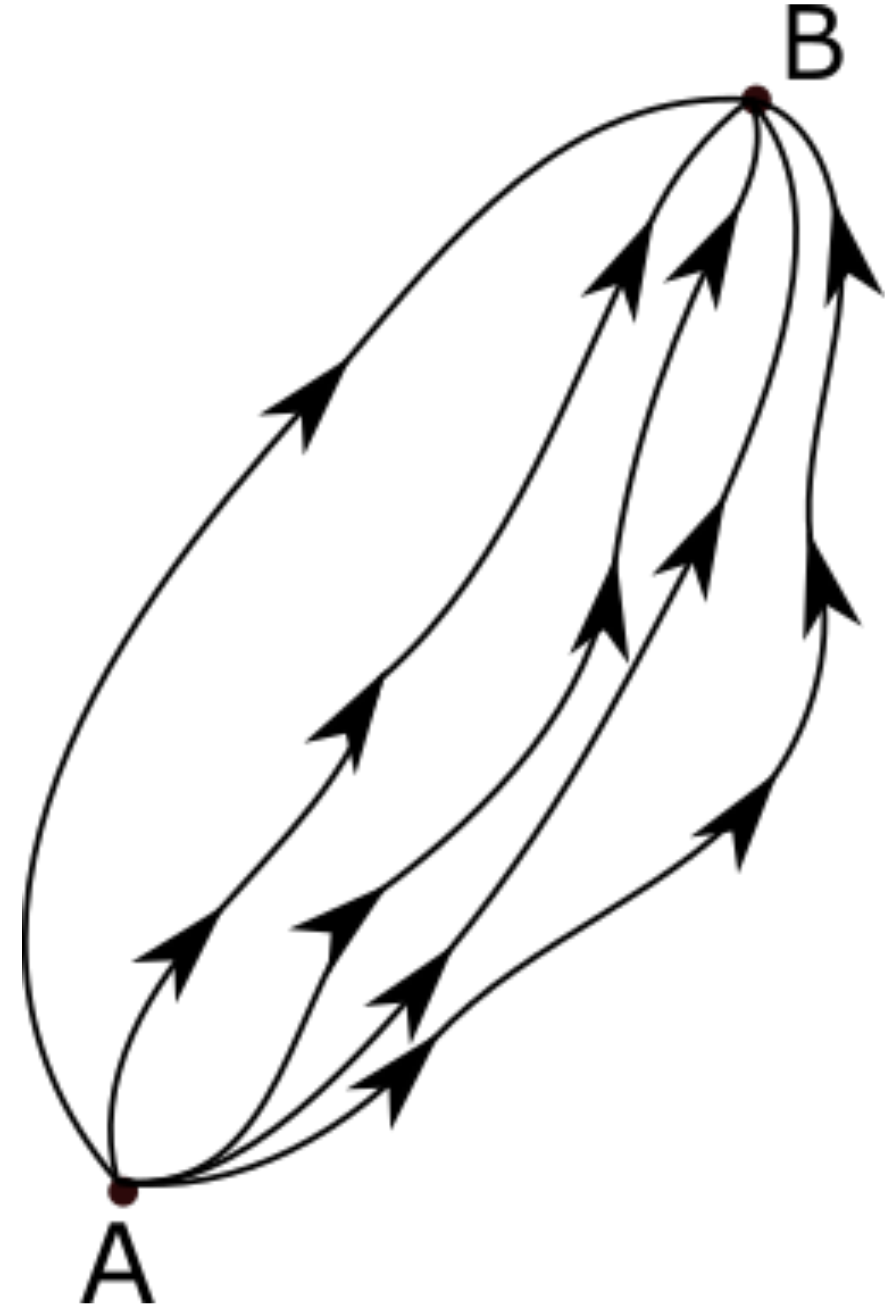

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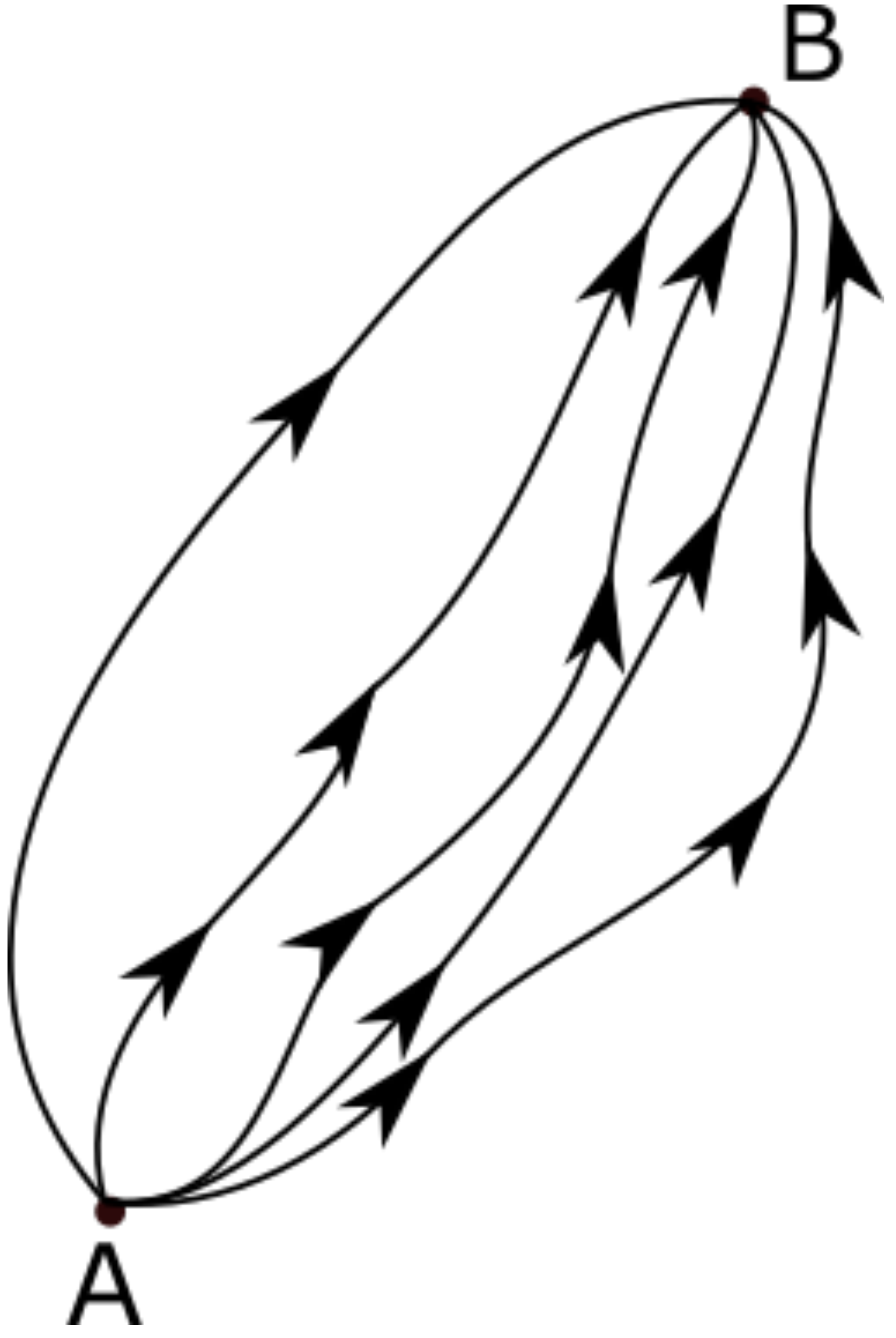
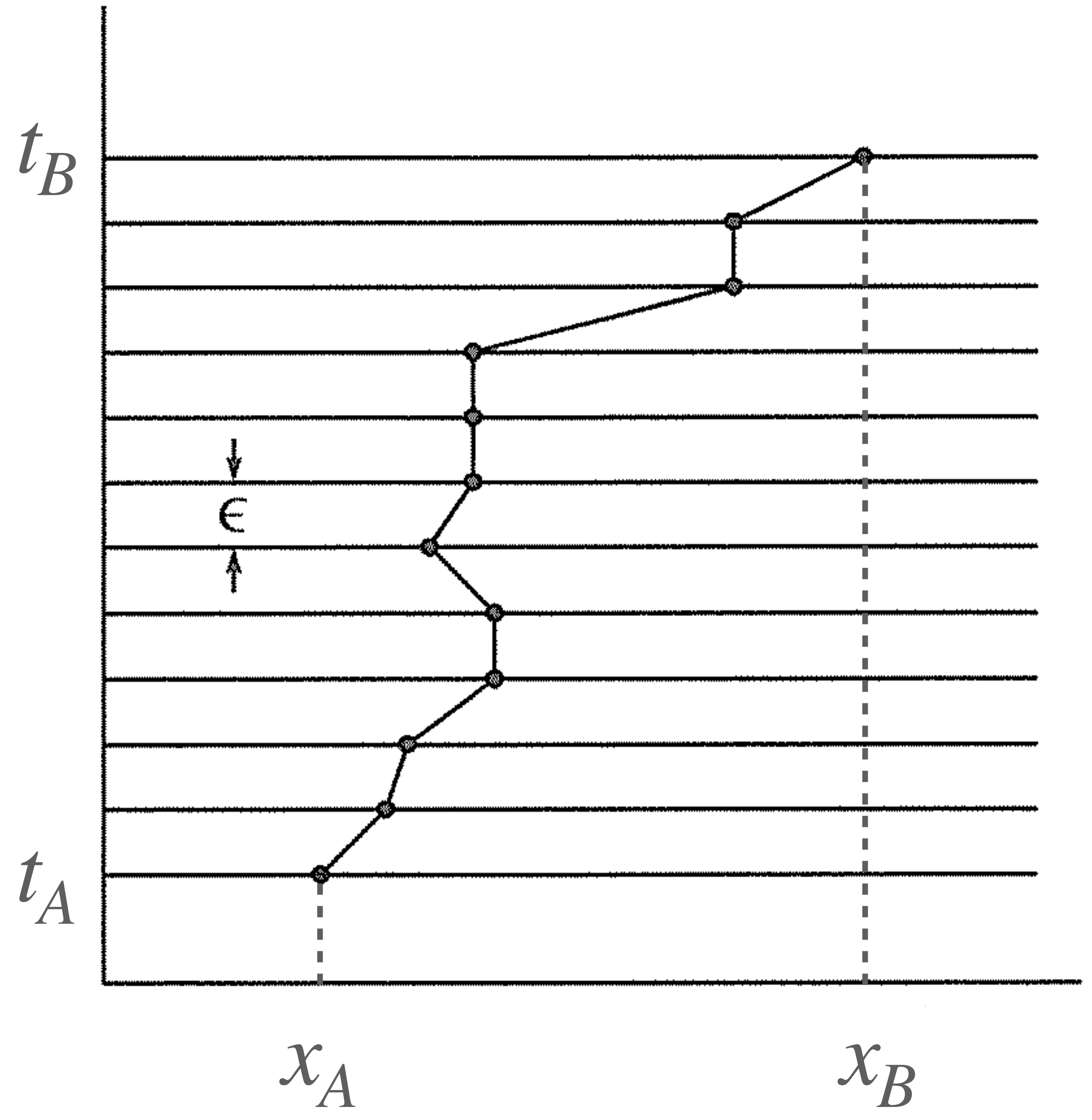
    - $\phi[x(t)] = (\text{const})e^{(i/\hbar)S[x(t)]}$
- This is a generalization of the classical principle of least action, sometimes called the ***quantum action principle***
    - Contains the classical principle in the limit $\hbar \to 0$ (or equivalently when $S \gg \hbar$



Defining the path integral



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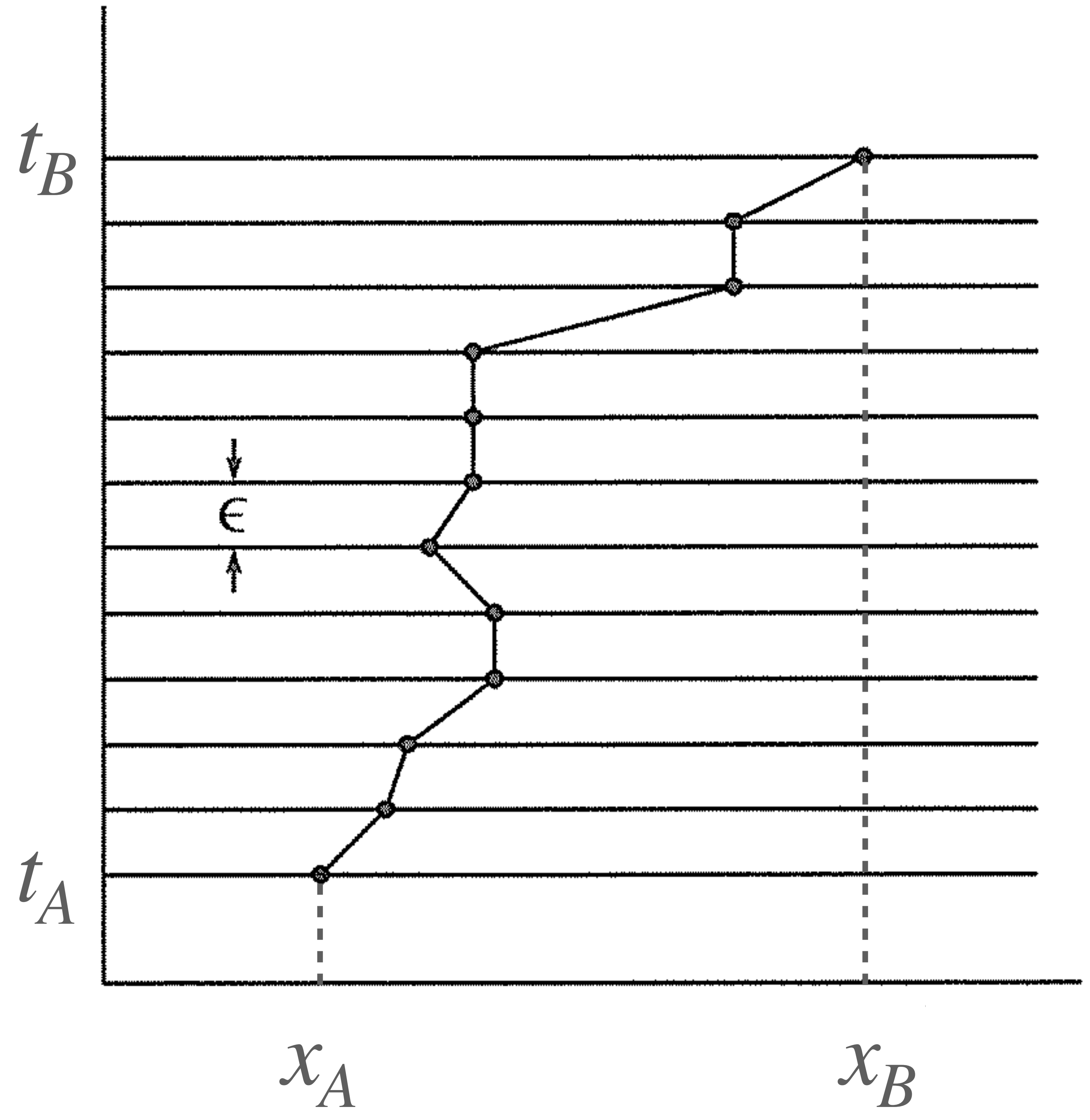
- First, choose a subset of all paths from (x_A, t_A) to (x_B, t_B) as follows



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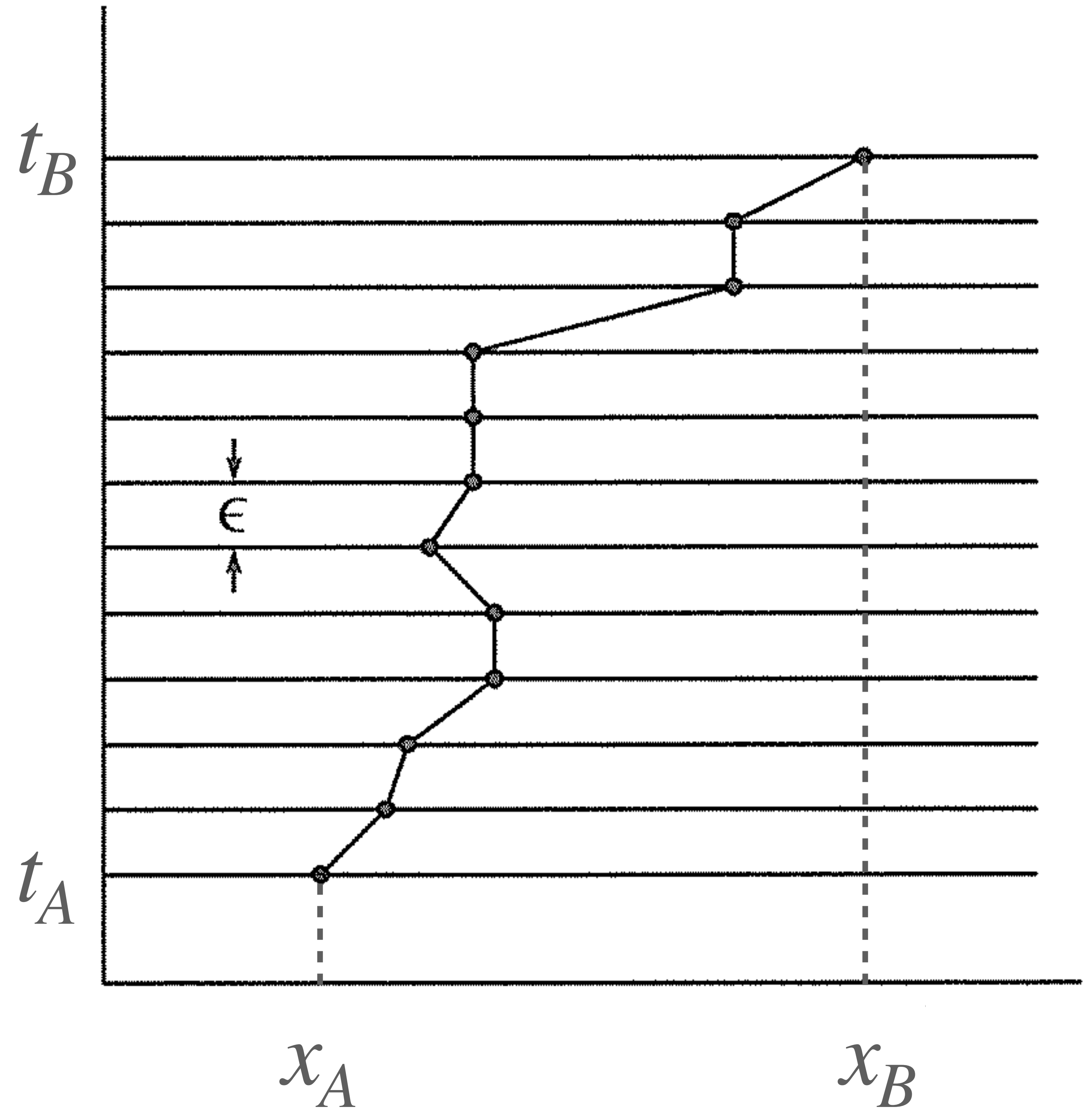

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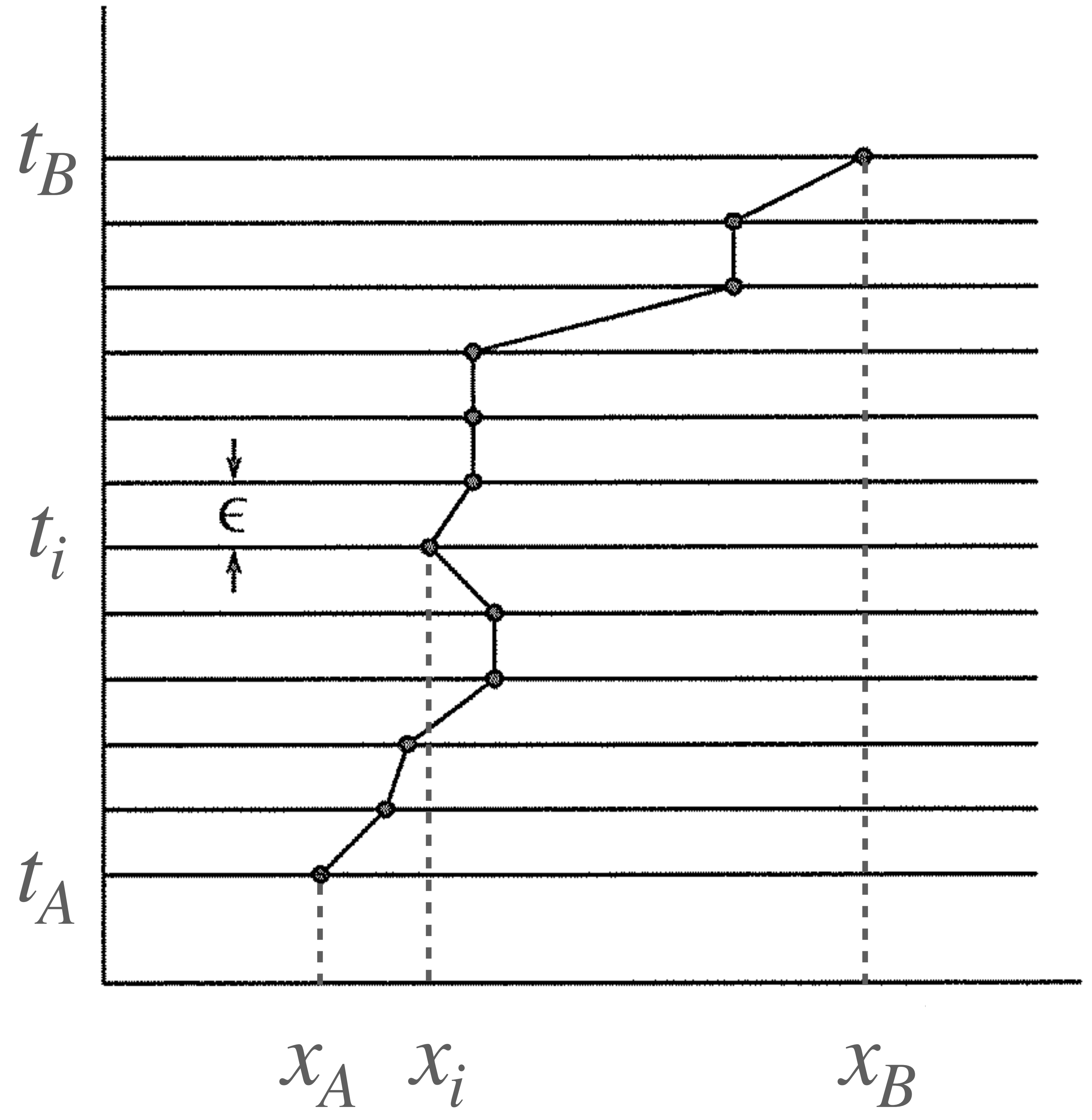
- At each time t_i, the path passes through some chosen point x_i



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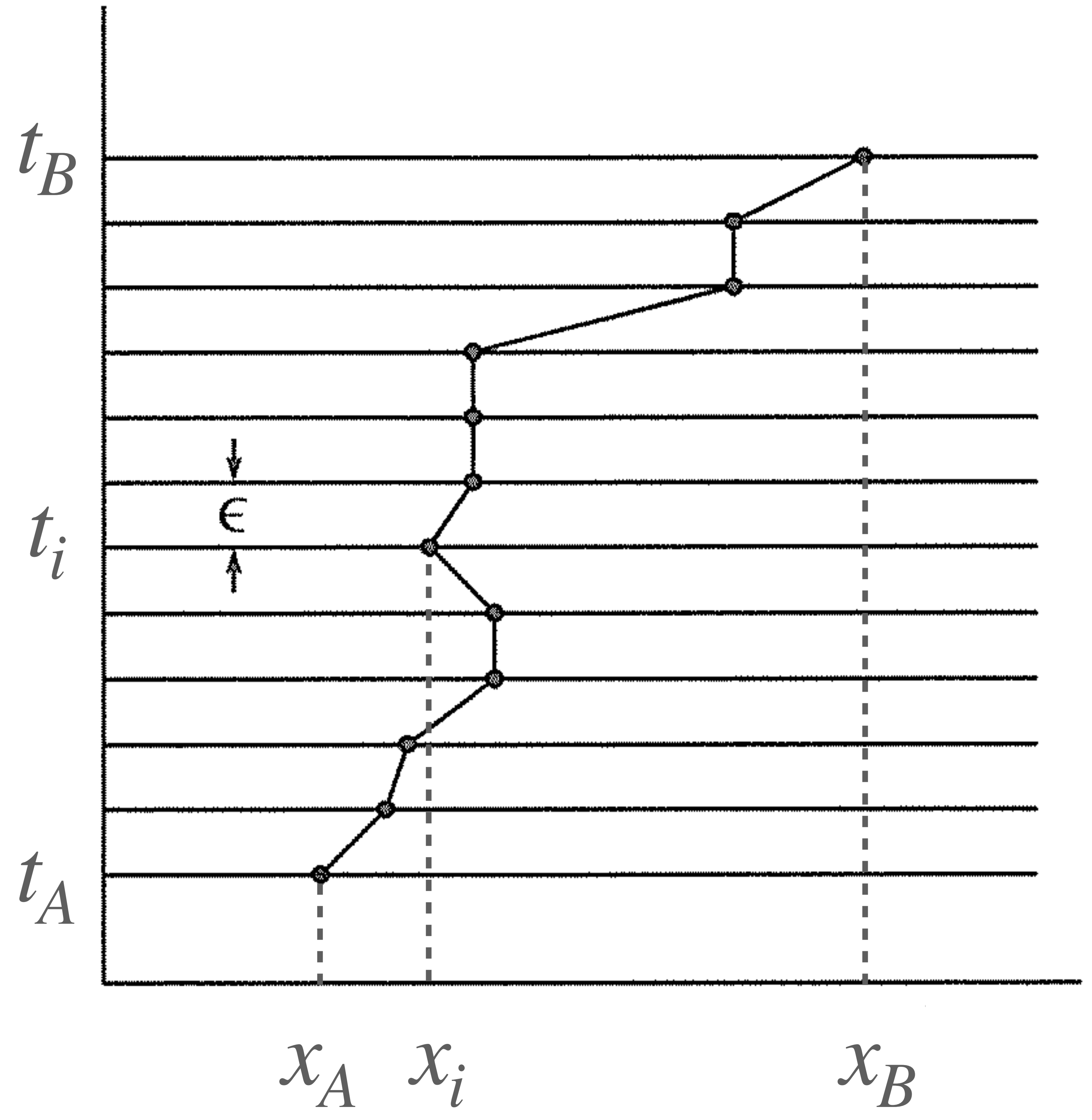
- We construct a path by connecting all the points with straight lines (i.e. constant velocity)



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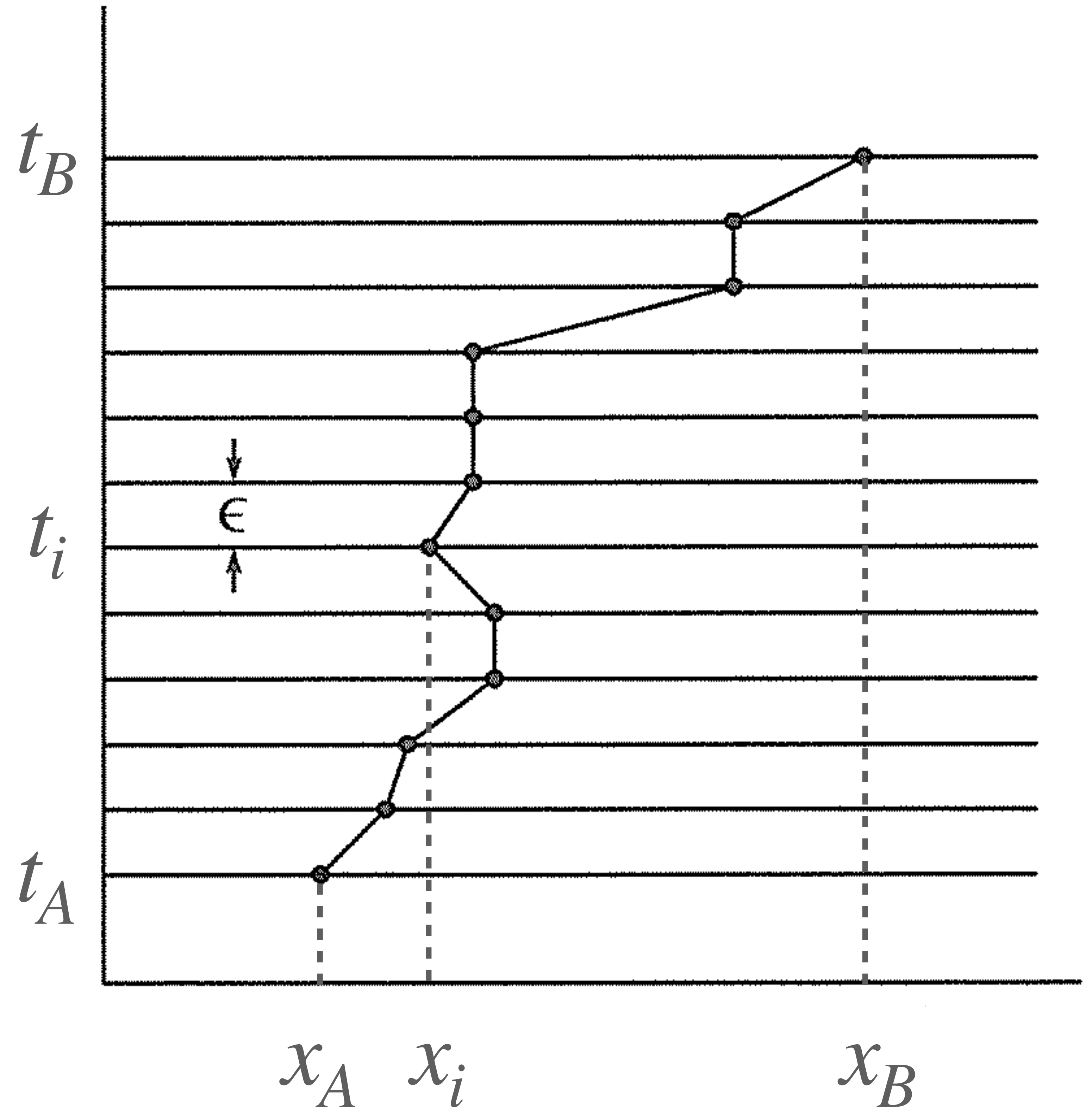

- Summary:

$$N\epsilon = t_B - t_A, \qquad \epsilon = t_{i+1} - t_i, \qquad t_0 = t_A, \qquad t_N = t_B$$
$$x_0 = x_A, \qquad x_N = x_B$$

Path integral for a free particle

Path integral for a free particle

- Path integral is $K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar} S[B,A]}$
- So we need to calculate the action for each discretized path from

  $S[B, A] = \int_{t_A}^{t_B} L(x, \dot{x}, t) dt$
- Note: we can get the action for the full path by adding up the contribution from each component of the path, i.e. assuming $t_A < t_C < t_B$, then the action along any path between A and B is
  - $S[B, A] = S[B, C] + S[C, A]$

Path integral for a free particle

Path integral for a free particle

Putting this together with the result of Quiz 1,

$$S[i, i-1] = \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}$$, and

$$S[B, A] = \sum_{i=1}^{N} S[i, i-1] = \sum_{i=1}^{N} \frac{m}{2}\frac{(x_i - x_{i-1})^2}{\epsilon}$$

So,

$$K(B, A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_N \left(\frac{2\pi i \hbar \epsilon}{m}\right)^{-\frac{N}{2}} \exp\left[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2\right]$$

Note this is a ***Gaussian integral***

Gaussian integrals

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More generally, $\int_{-\infty}^{+\infty} e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)+c}$

With complex arguments, $\int_{-\infty}^{+\infty} e^{iax^2+ibx+c} = \sqrt{\frac{i\pi}{a}} e^{-ib^2/(4a)+c}$

Recursive pattern

Recursive pattern

A pattern emerges… After $N-1$ steps, we get

$$K(B,A) = \lim_{\epsilon\to 0} \int \cdots \int dx_1 \cdots dx_N \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-\frac{N}{2}} \exp\left[\frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(x_i - x_{i-1})^2\right]$$

$$= \lim_{\epsilon\to 0} \left(\frac{m}{2\pi i\hbar(N\epsilon)}\right)^{1/2} \exp\left[\frac{im}{2\hbar(N\epsilon)}(x_N - x_0)^2\right]$$

$$= \left(\frac{m}{2\pi i\hbar(t_B - t_A)}\right)^{1/2} \exp\left[\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)}\right]$$

because $N\epsilon = t_B - t_A, x_0 = x_A, x_N = x_B$

Free particle behavior

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The free particle propagator encodes a lot of ***physics***

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For convenience, let's analyze it when $A = (0,0)$ and $B = (x, t)$; i.e. a particle is traveling from the origin to position x in time t

$$K(x, t; 0,0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{imx^2}{2\hbar t}\right]$$

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Then let's do the same at fixed x, for the propagator as a function of time t

Reminder of plane waves

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Plane wave solution to 1D Schrödinger equation

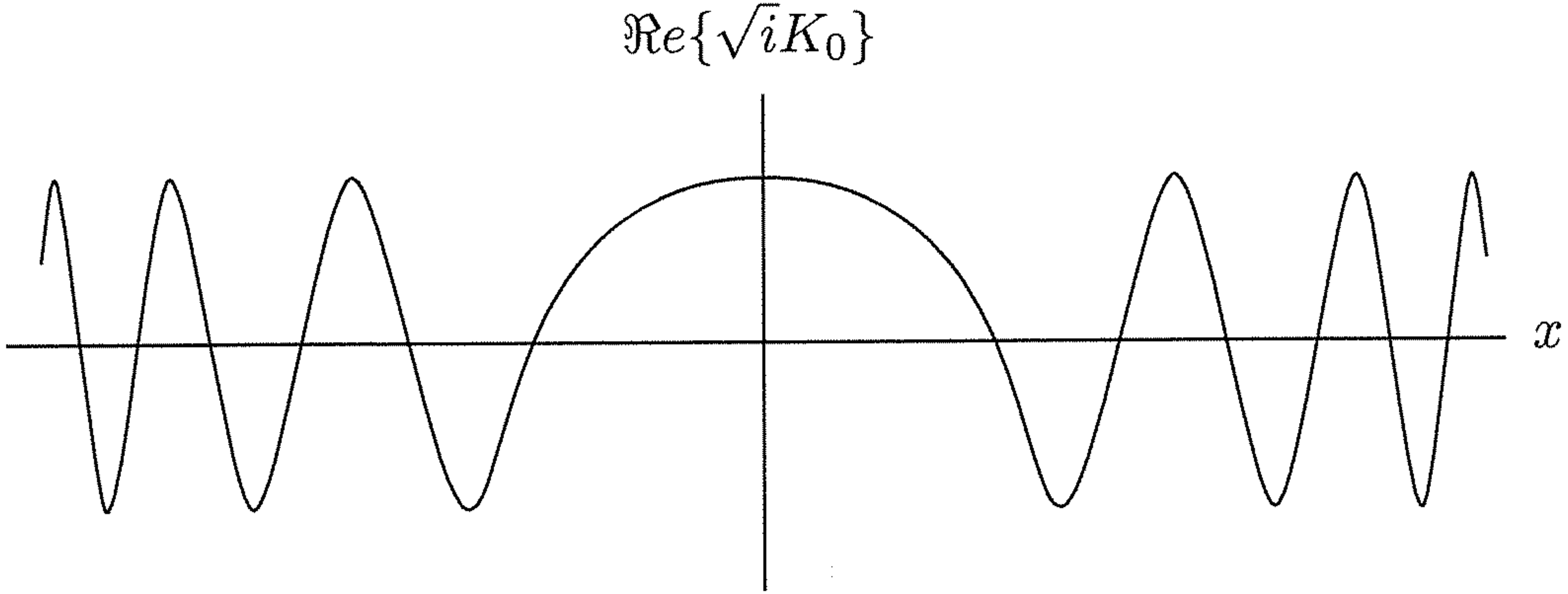
Reminder of plane waves

Plane wave solution to 1D Schrödinger equation

$$\psi(x,t) = C\exp[i(kx - \omega t)]$$

means particle has momentum $p = \hbar k$ and energy $E = \hbar\omega$

Free particle behavior at large x



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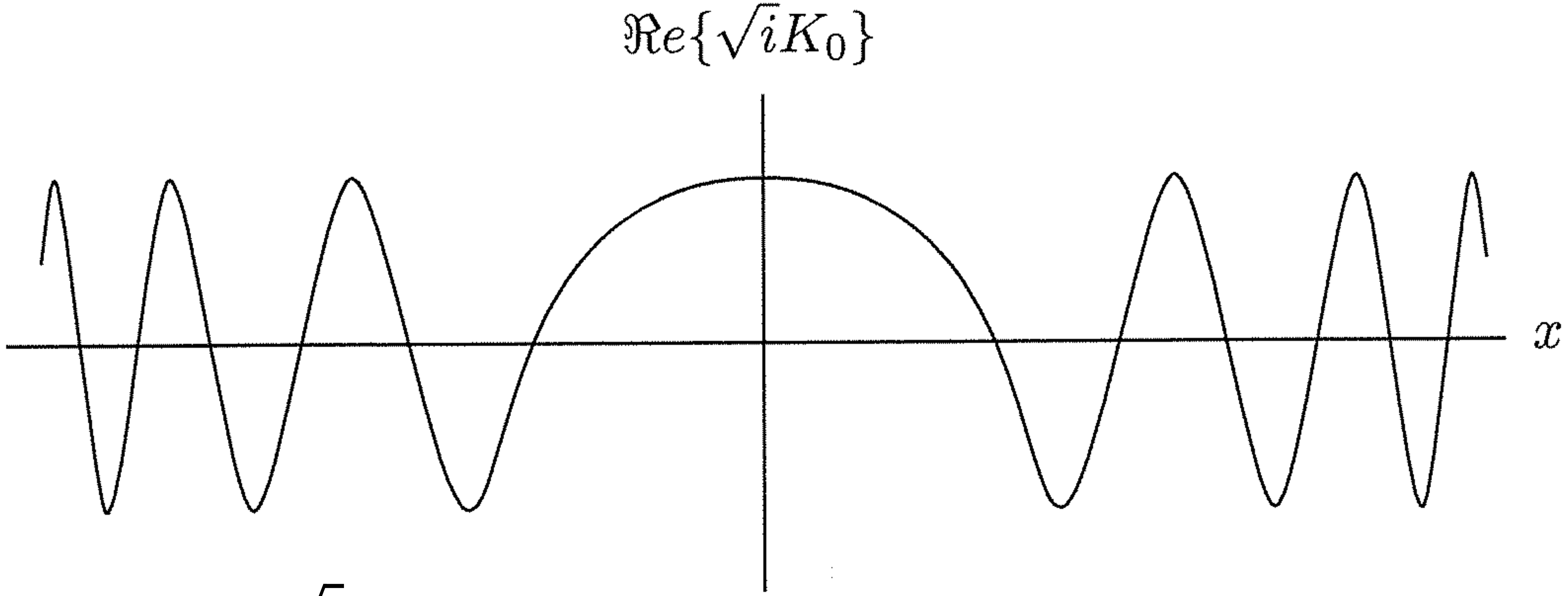

At large x (fixed t), $\Re[\sqrt{i}K]$ oscillates with slowly varying wavelength λ

$$2\pi = \frac{m(x+\lambda)^2}{2\hbar t} - \frac{mx^2}{2\hbar t} = \frac{mx\lambda}{\hbar t} + \frac{m\lambda^2}{2\hbar t} \Rightarrow \lambda \approx \frac{2\pi\hbar}{m(x/t)}$$

assuming $x \gg \lambda$

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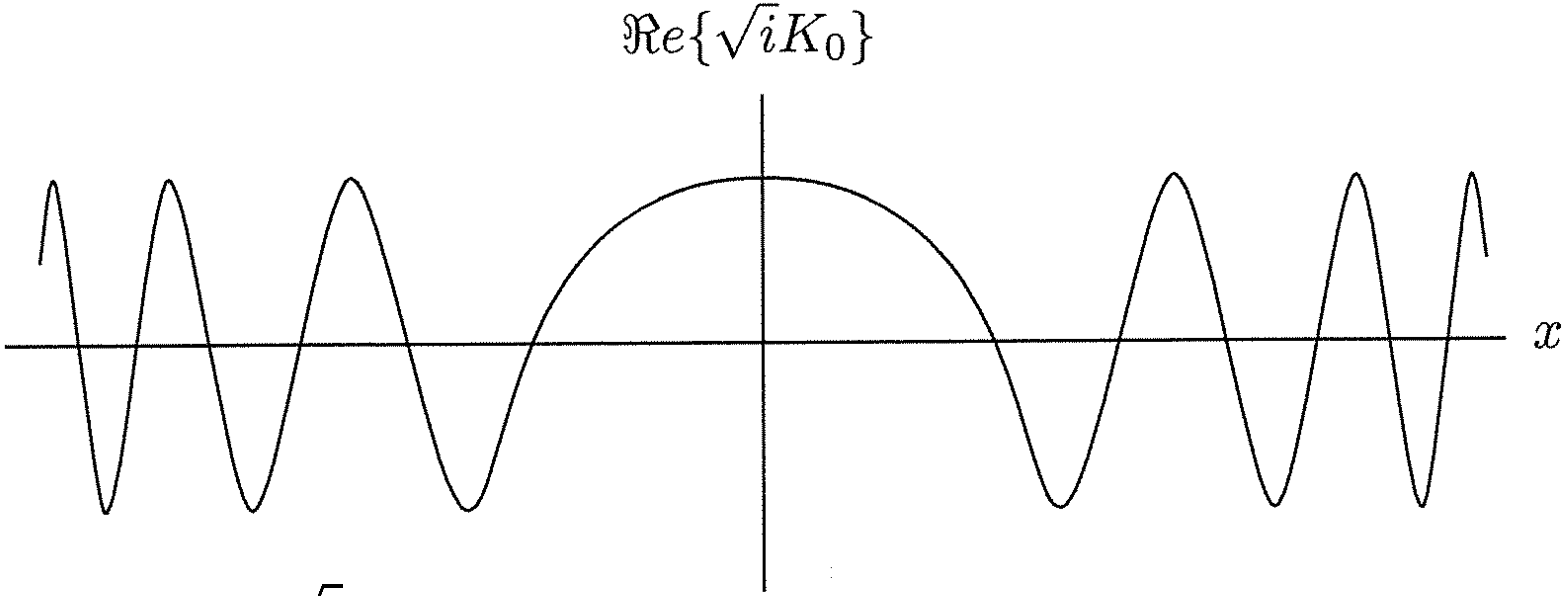

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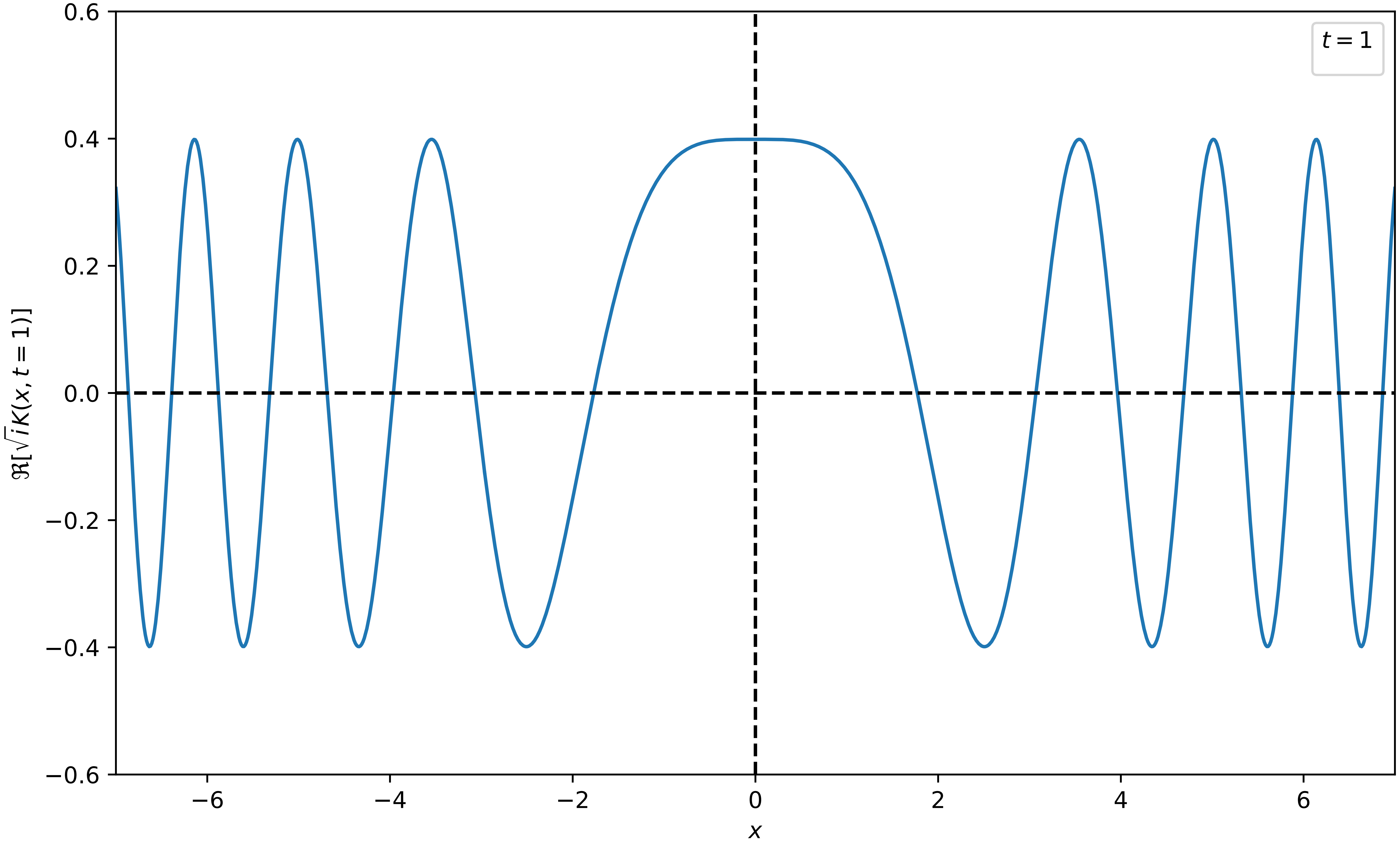
$k = \dfrac{2\pi}{\lambda} = \dfrac{m(x/t)}{\hbar}$ and $p = \hbar k$!

Compare to plane wave at $(x = 5, t = 1)$



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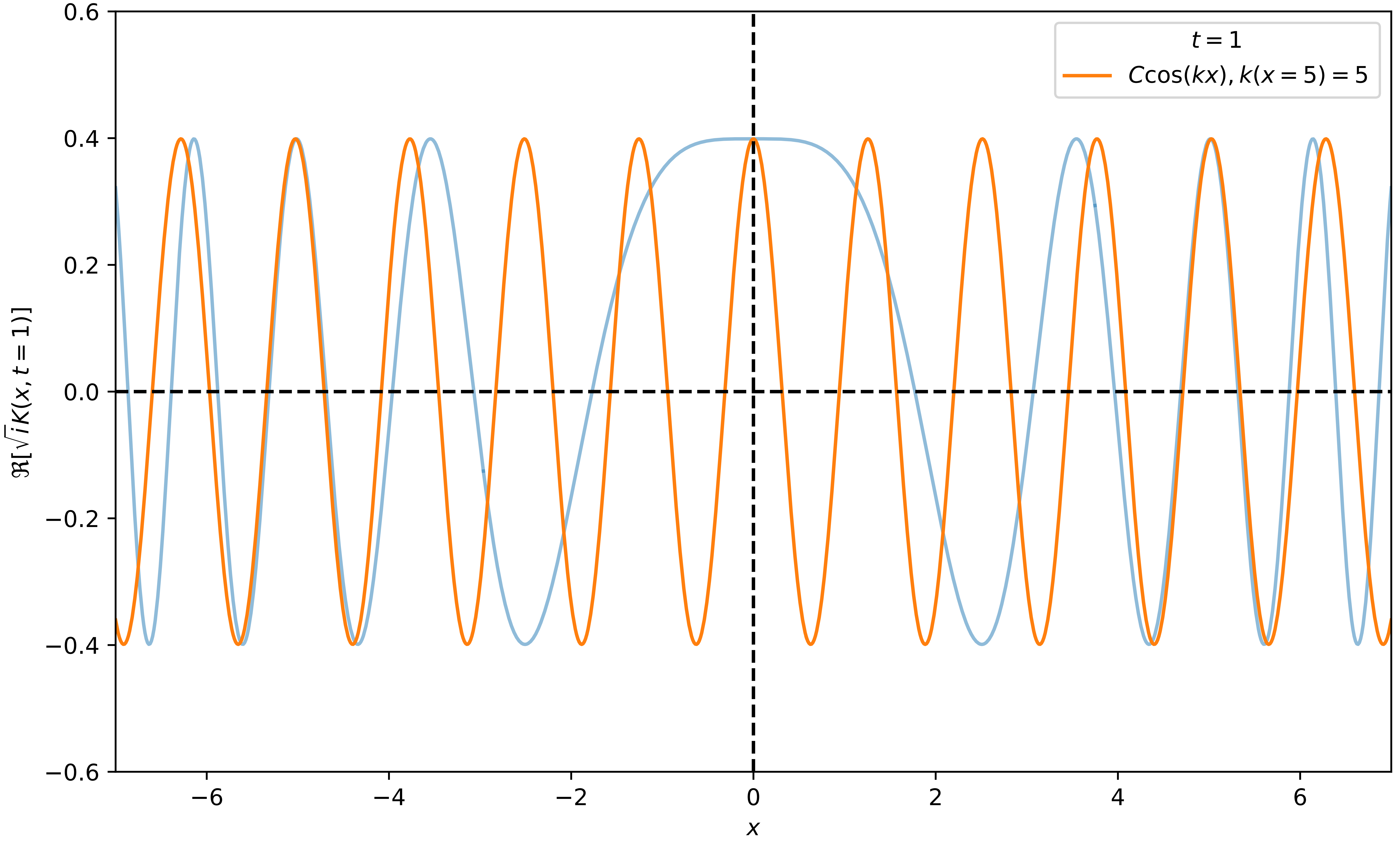
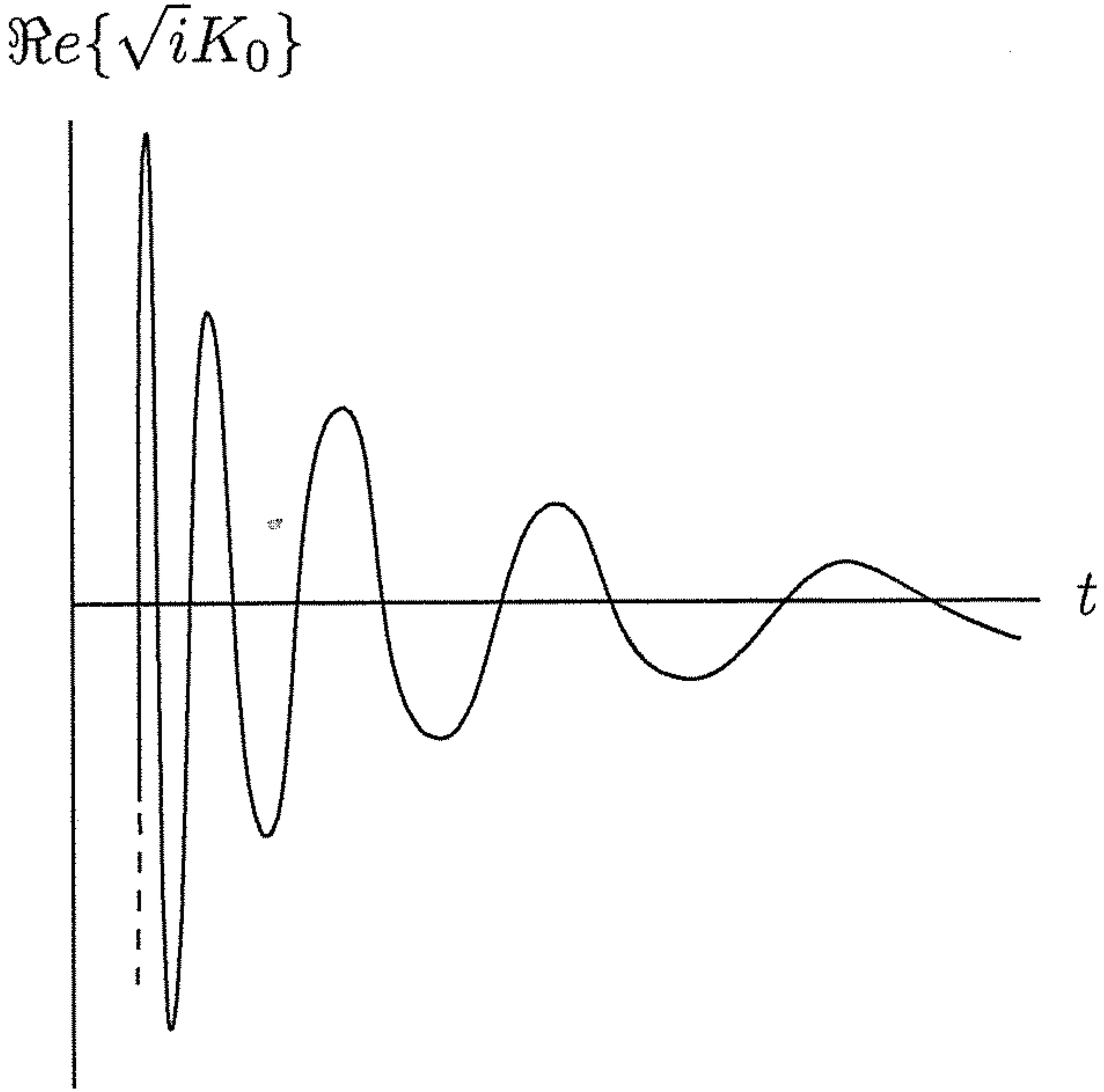


Free particle behavior at large t



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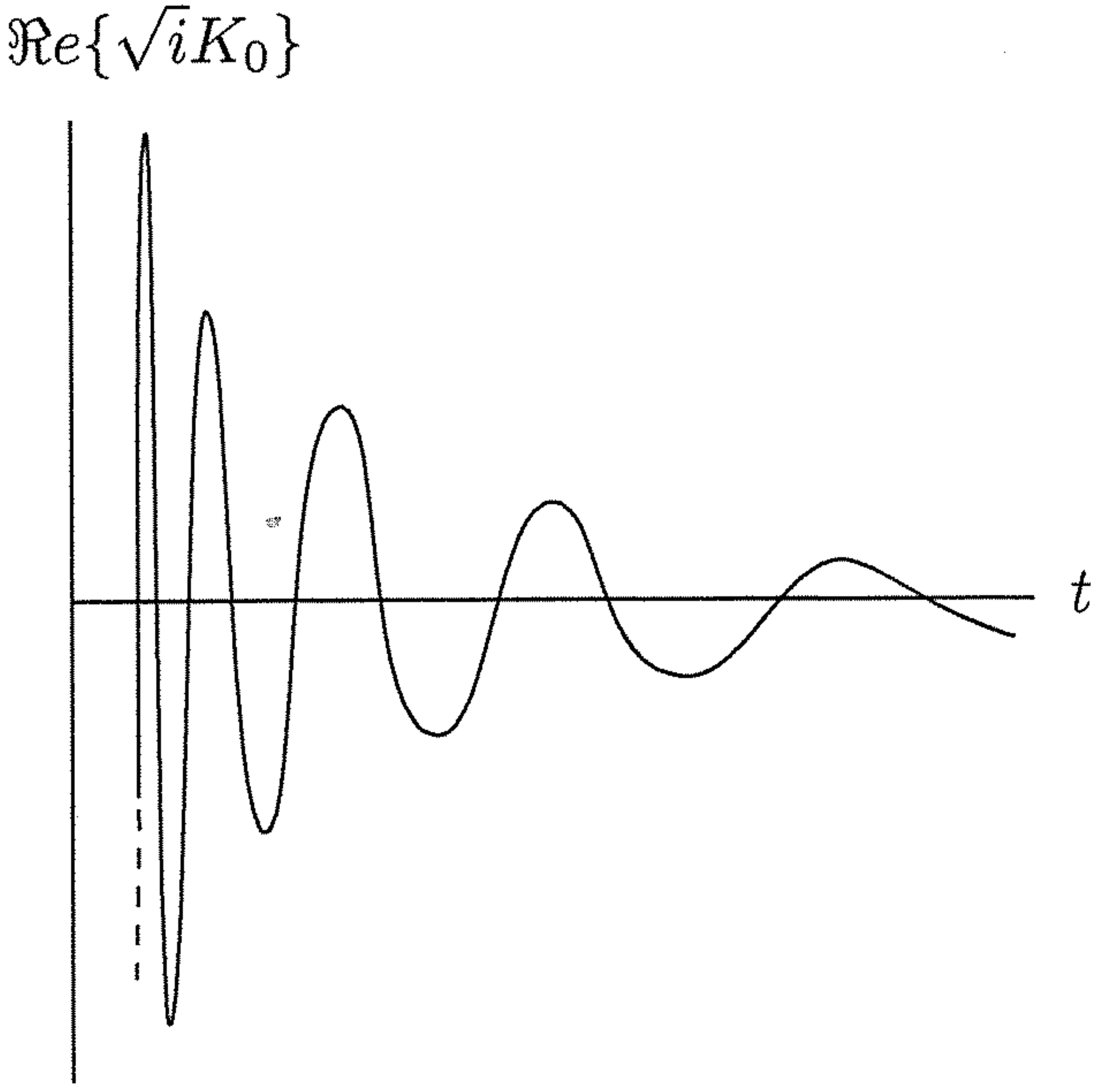


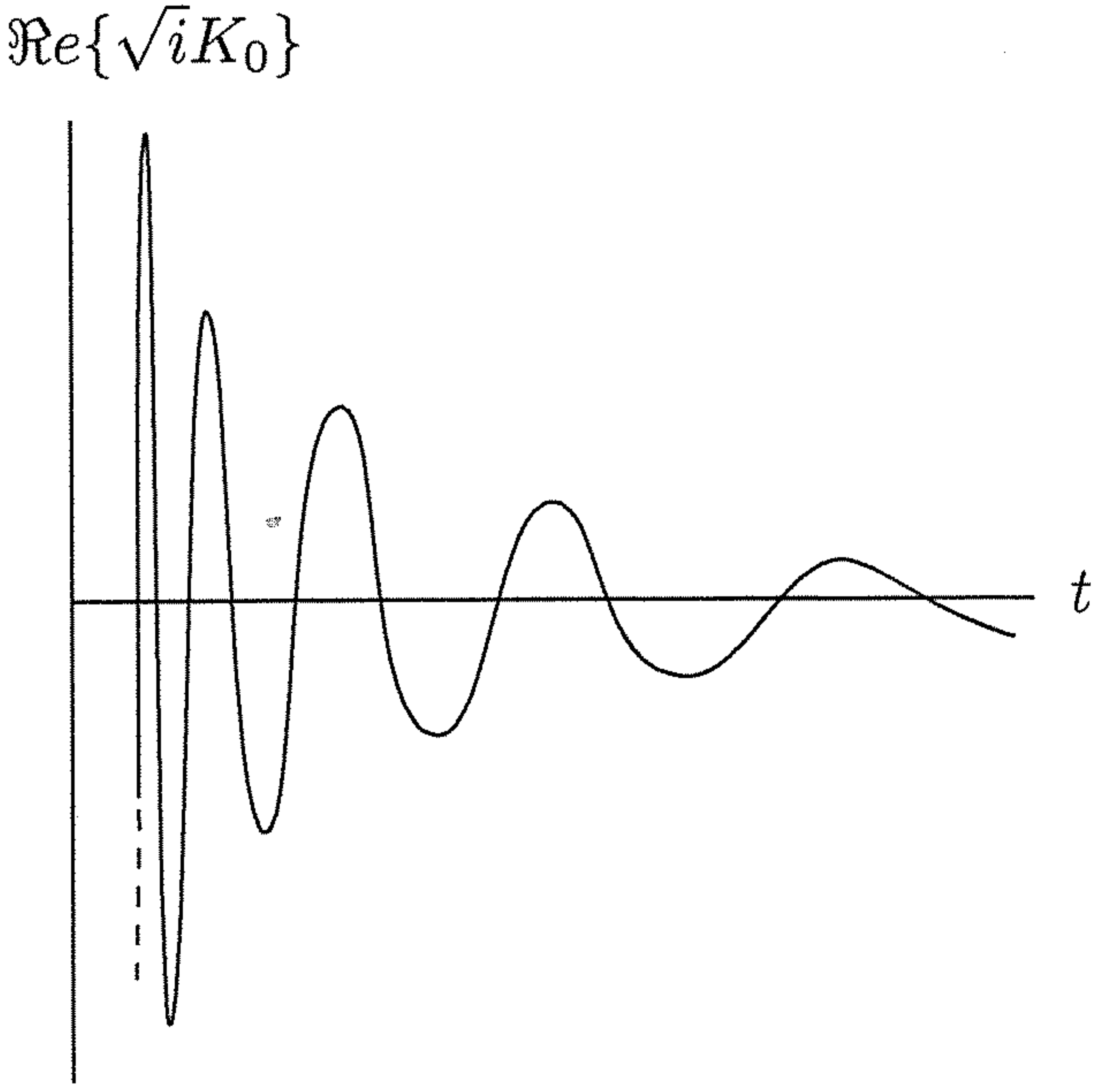
At large t (fixed x), $\mathfrak{R}[\sqrt{i}K]$ amplitude and period varies; Neglecting change in amplitude,

$$2\pi = \frac{mx^2}{2\hbar t} - \frac{mx^2}{2\hbar(t+T)} = \frac{mx^2}{2\hbar t}\left(\frac{T}{1+T/t}\right)$$

$$\approx \frac{mT}{2\hbar}\left(\frac{x}{t}\right)^2 \Rightarrow T \approx \frac{\pi\hbar}{m(x/t)^2}$$

assuming $t \gg T$

Free particle behavior at large t



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$$\approx \frac{mT}{2\hbar}\left(\frac{x}{t}\right)^2 \Rightarrow T \approx \frac{\pi\hbar}{m(x/t)^2}$$

assuming $t \gg T$

$$\omega = \frac{2\pi}{T} = \frac{m}{2\hbar}\left(\frac{x}{t}\right)^2 \text{ and } E = \hbar\omega \,!$$

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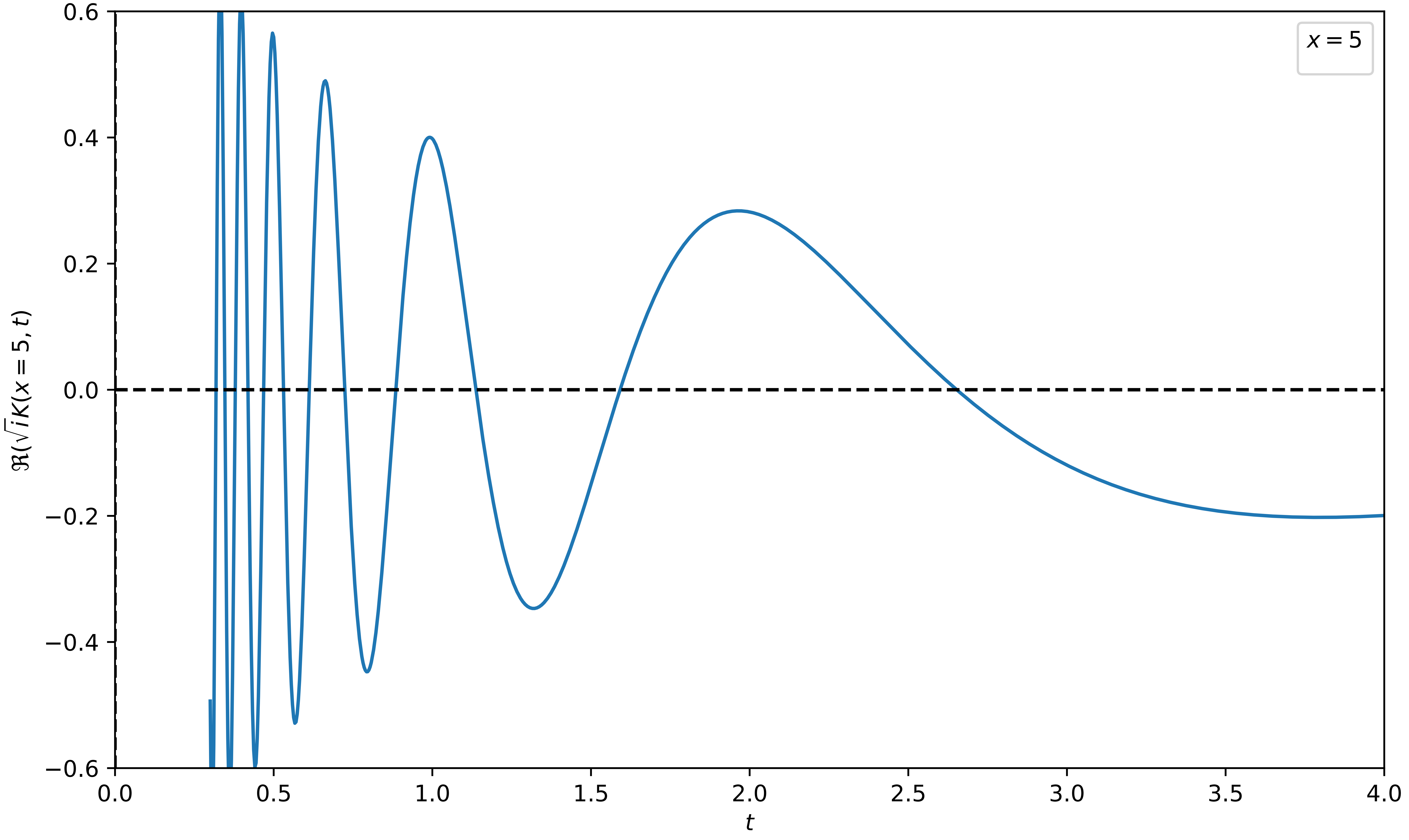

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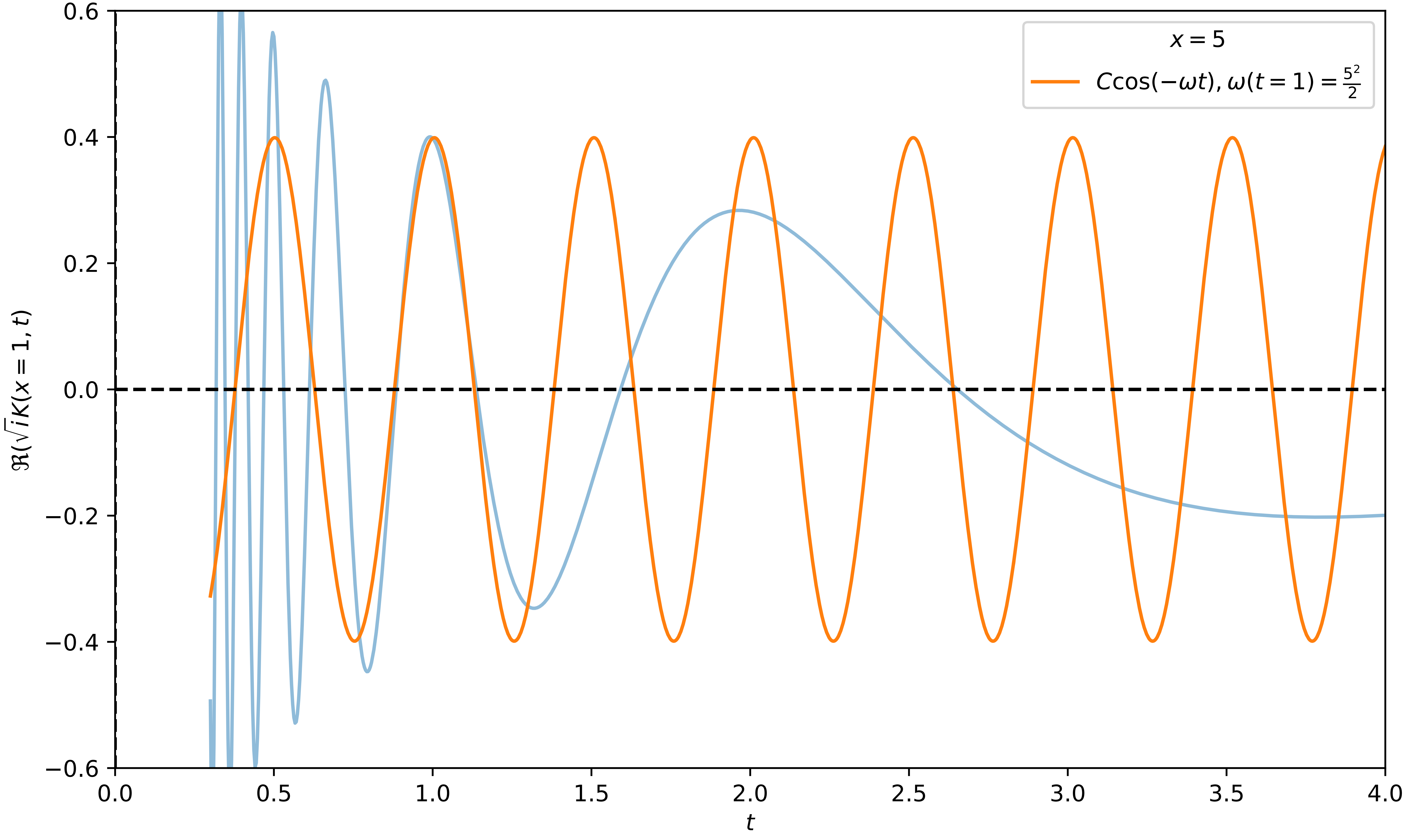

Energy and momentum and wave

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Concepts of momentum and energy are extended to quantum mechanics as expected

1. If the amplitude varies in space as e^{ikx}, we say that the particle has momentum $\hbar k$

2. If the amplitude varies in time as $e^{-i\omega t}$, we say that the particle has energy $\hbar\omega$

Wave functions and Schrödinger Equation

Wave functions and Schrödinger Equation

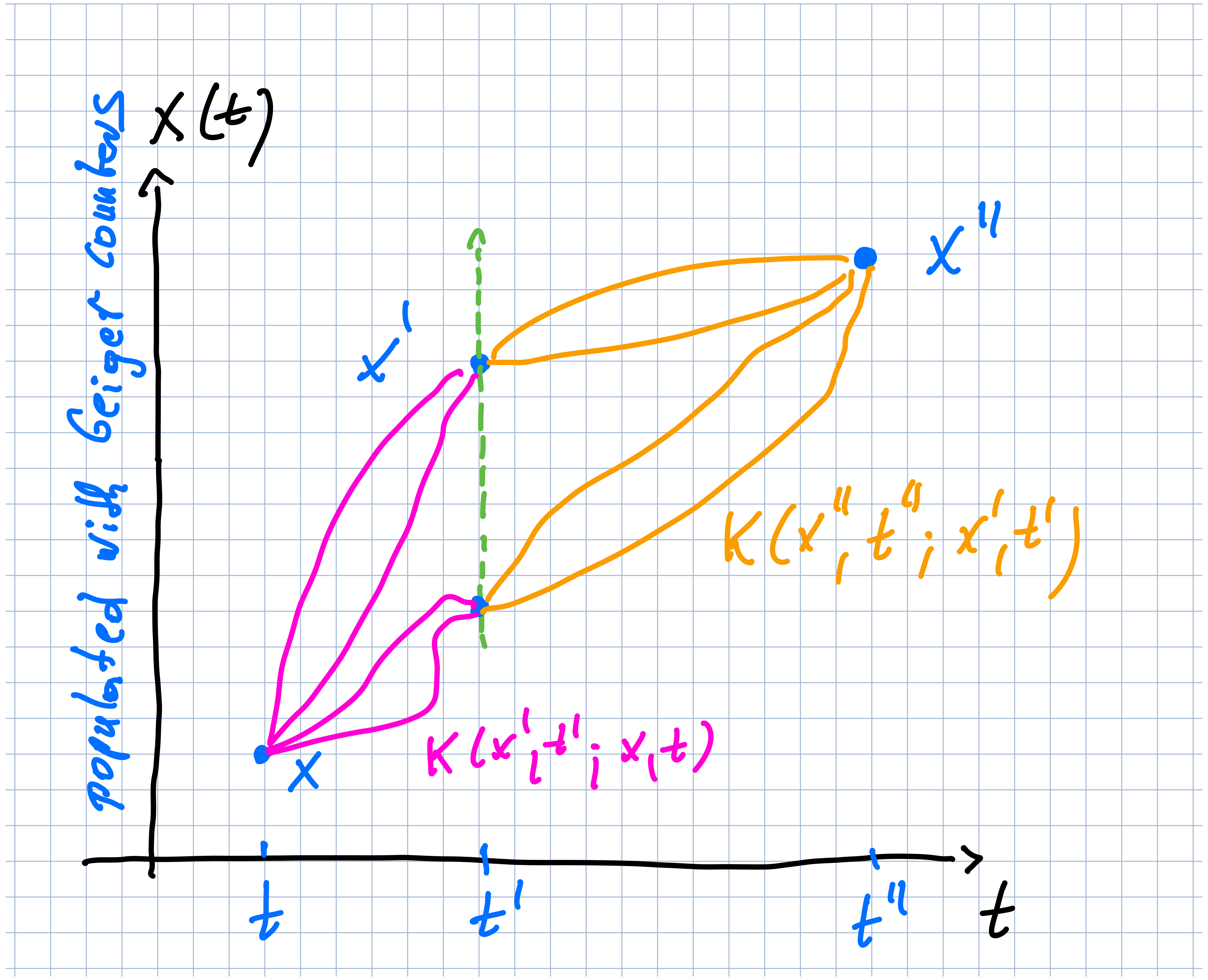
The propagator can tell us the probability amplitude to go from one state (described by a wave function) to another

$$\psi(x', t') = \int_{-\infty}^{\infty} K(x', t'; x, t)\psi_0(x, t)dx$$

In Quiz 2, you will show that the free propagator satisfies the Schrödinger equation!

$$-\frac{\hbar}{i}\frac{\partial K(B,A)}{\partial t_B} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2 K(B,A)}{\partial x_B^2}\right] \text{ for } t_B > t_A$$

Chaining Propagators



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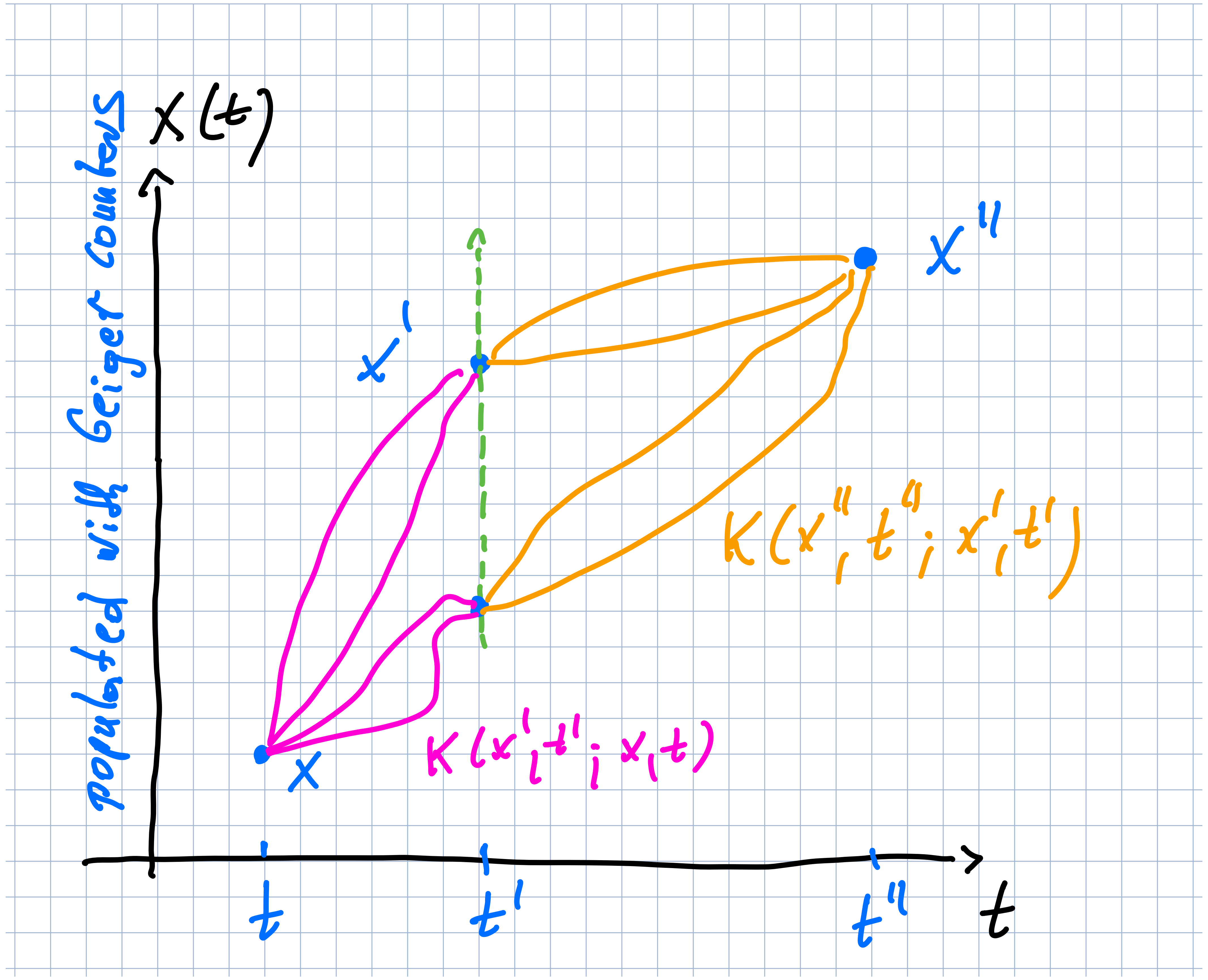

$$\widetilde{\psi}(x'', t'') = \int_{-\infty}^{\infty} K(x'', t''; x', t')\psi(x', t')dx'$$

$$\psi(x', t') = \int_{-\infty}^{\infty} K(x', t'; x, t)\psi_0(x, t)dx$$

$$\Rightarrow \widetilde{\psi}(x'', t'') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x'', t''; x', t')K(x', t'; x, t)\psi_0(x, t)dx'dx$$