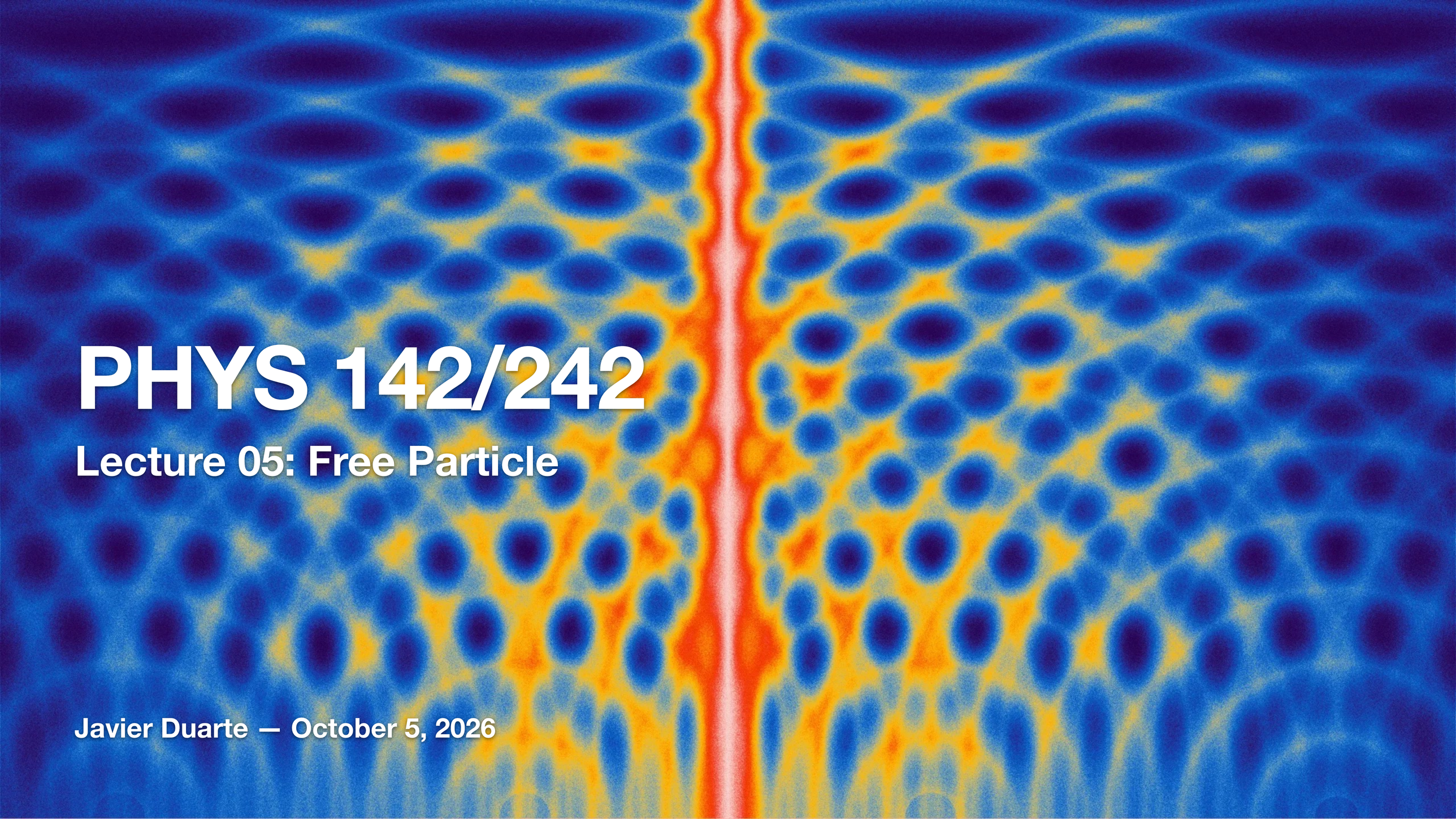
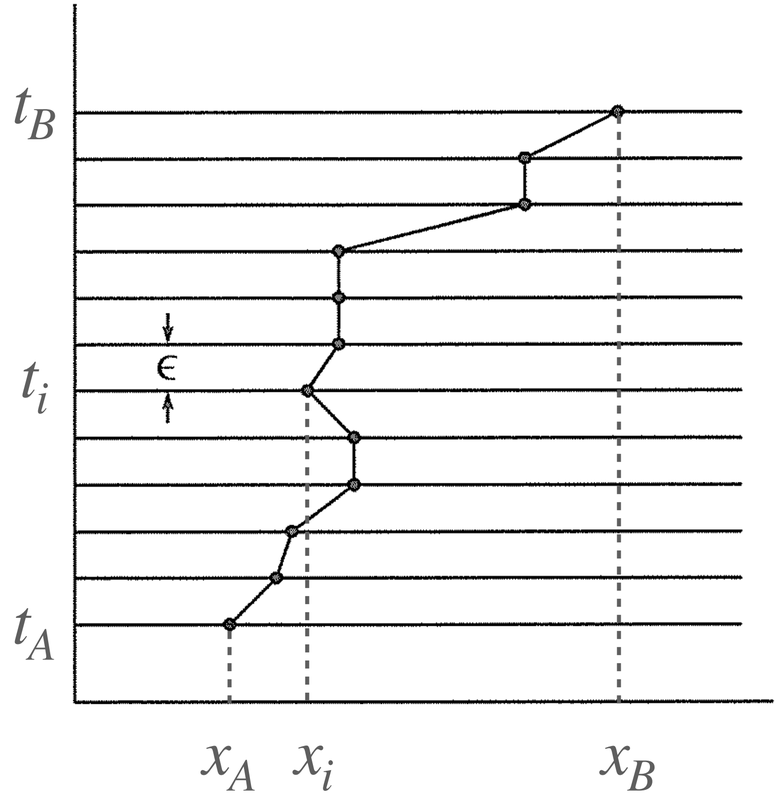


PHYS 142/242

Lecture 05: Free Particle

Javier Duarte — October 5, 2026

Path integral



The path integral is

$$K(x_B, t_B; x_A, t_A) = \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S[B,A]}$$

$$= \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar}S[B,A]}$$

Path integral for a free particle

Last time: solved the path integral for a free particle

$$K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} \exp \left[\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2 \right]$$

$$= \left(\frac{m}{2\pi i \hbar (t_B - t_A)} \right)^{1/2} \exp \left[\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)} \right]$$

Today: Briefly examine its behavior to see if it's consistent with a particle with

momentum $m(x_B - x_A)/(t_B - t_A)$ and

energy $\frac{1}{2} m((x_B - x_A)/(t_B - t_A))^2$

Free particle behavior

For convenience, let $A = (0,0)$ and $B = (x, t)$; i.e. a particle is traveling from the origin to position x in time t

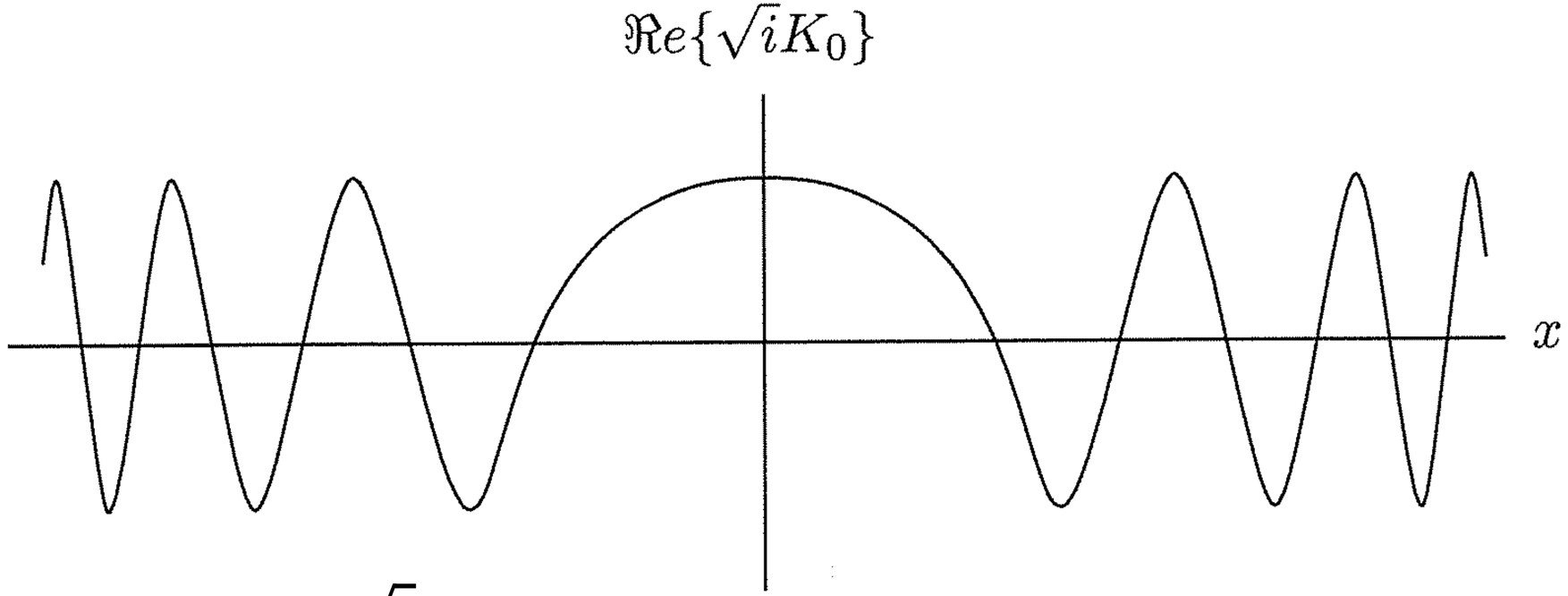
$$\psi(x, t) \propto K(x, t; 0,0) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left[\frac{imx^2}{2\hbar t}\right]$$

At fixed t, consider the propagator as a function of position x

Then at fixed x, consider the propagator as a function of time t

Note: if $\psi(x, t) \propto \exp[i(kx - \omega t)]$ then particle has
momentum $p = \hbar k$ and
energy $E = \hbar\omega$

Free particle behavior at large x

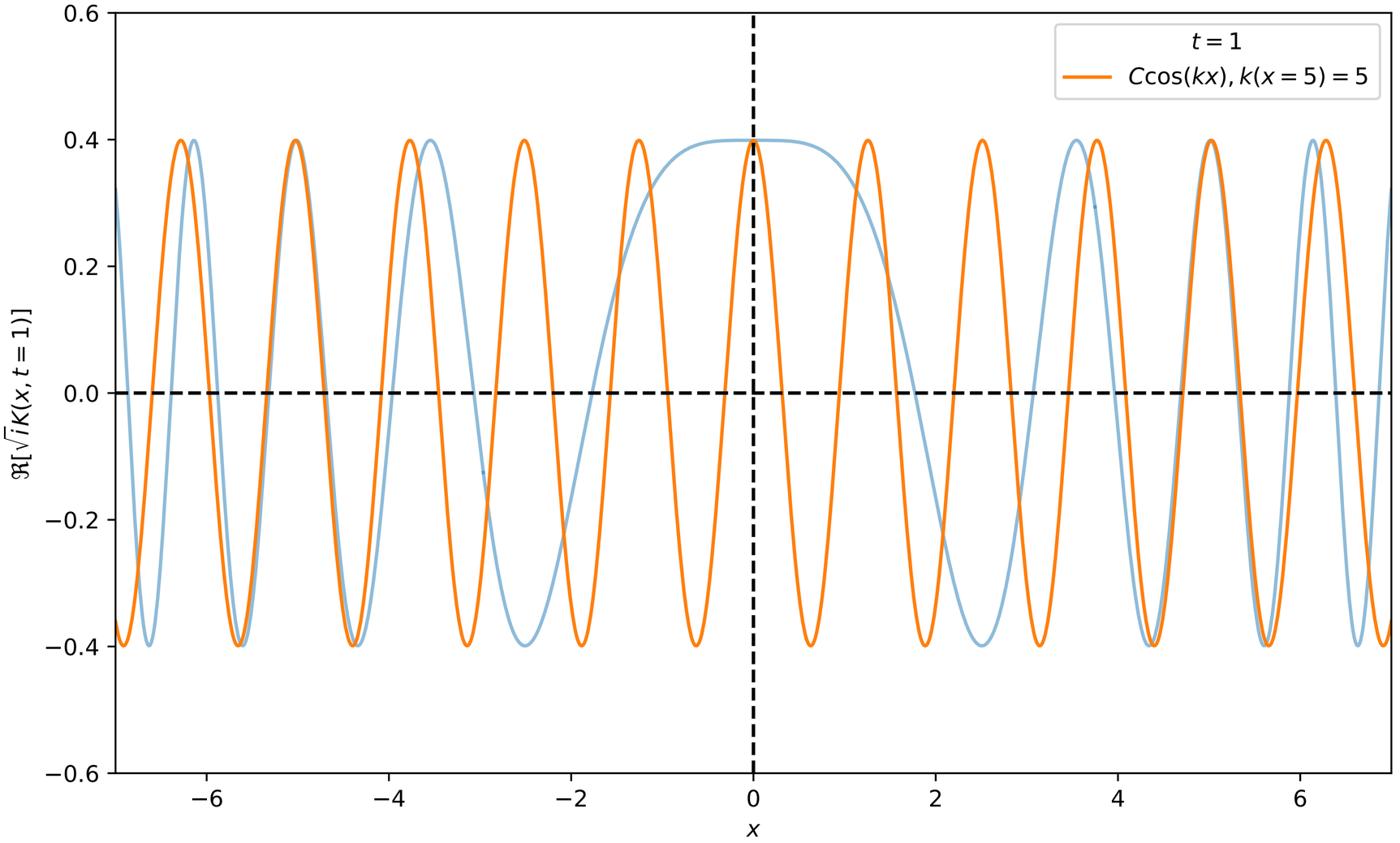


At large x (fixed t), $\Re[\sqrt{i}K]$ oscillates with slowly varying wavelength λ

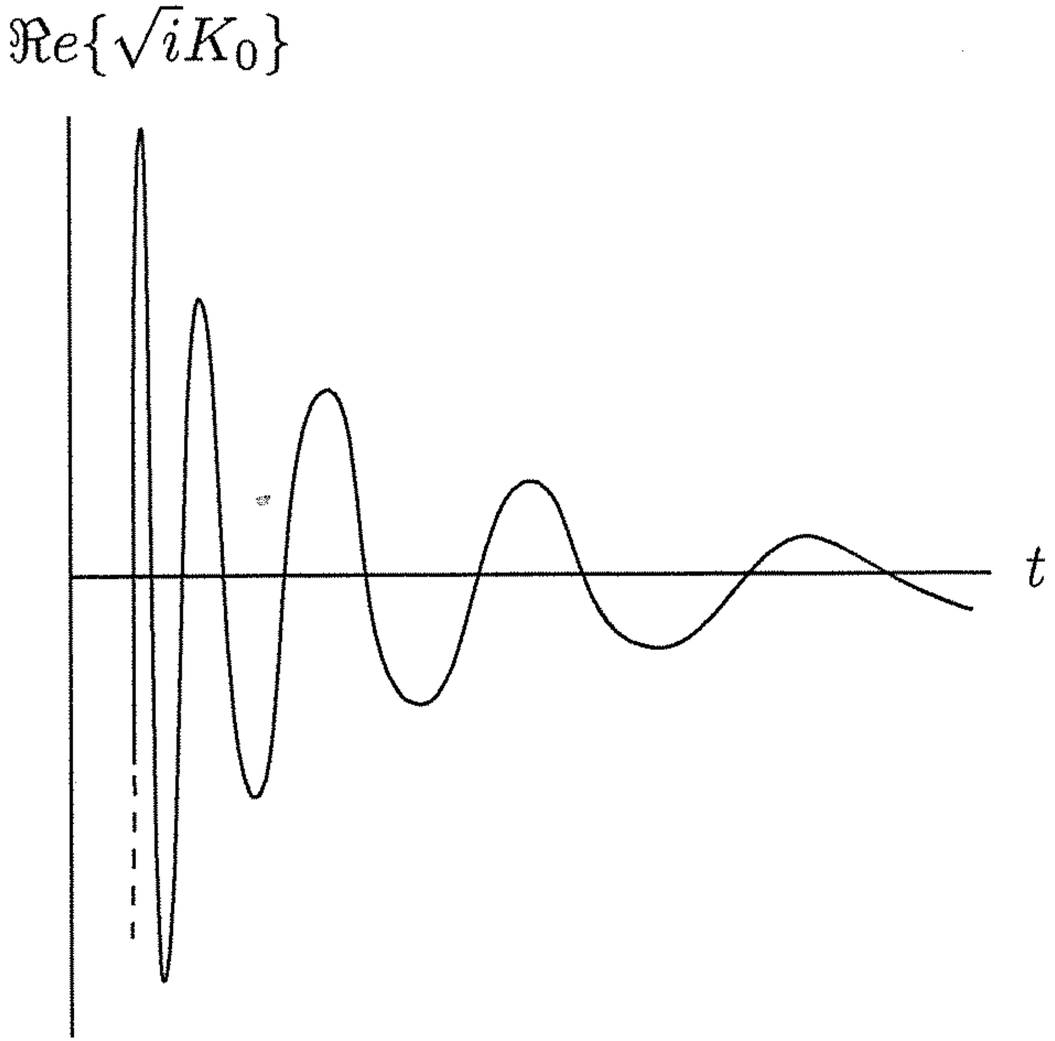
$$2\pi = \frac{m(x+\lambda)^2}{2\hbar t} - \frac{mx^2}{2\hbar t} = \frac{mx\lambda}{\hbar t} + \frac{m\lambda^2}{2\hbar t} \Rightarrow \lambda \approx \frac{2\pi\hbar}{m(x/t)}$$ assuming $x \gg \lambda$

$$k = \frac{2\pi}{\lambda} = \frac{m(x/t)}{\hbar}$$ and $p = \hbar k$!

Compare to plane wave at $(x = 5, t = 1)$



Free particle behavior at large t



At large t (fixed x), $\Re[\sqrt{i}K]$ amplitude and period varies; Neglecting change in amplitude,

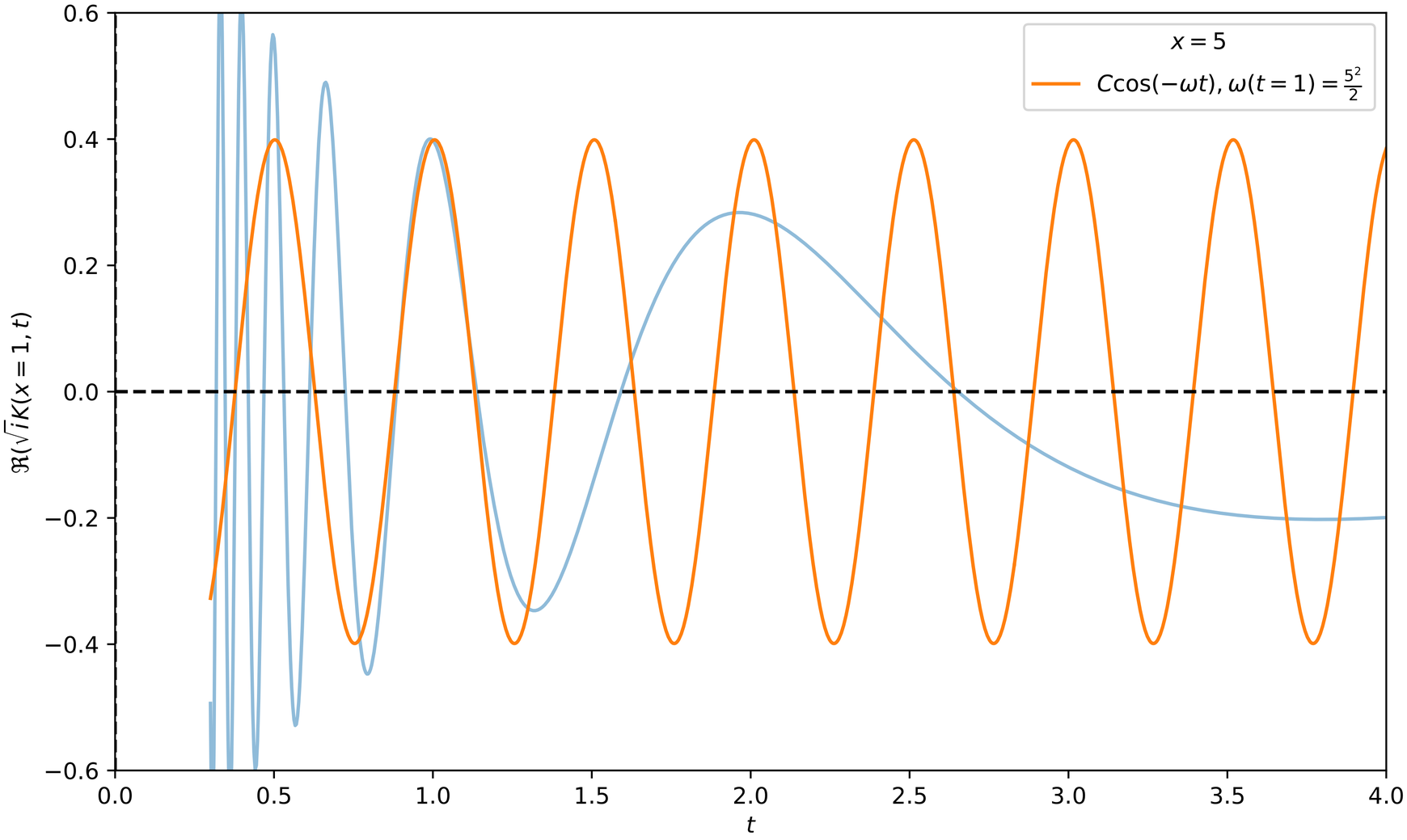
$$2\pi = \frac{mx^2}{2\hbar t} - \frac{mx^2}{2\hbar(t+T)} = \frac{mx^2}{2\hbar t}\left(\frac{T}{1+T/t}\right)$$

$$\approx \frac{mT}{2\hbar}\left(\frac{x}{t}\right)^2 \Rightarrow T \approx \frac{\pi\hbar}{m(x/t)^2}$$

assuming $t \gg T$

$$\omega = \frac{2\pi}{T} = \frac{m}{2\hbar}\left(\frac{x}{t}\right)^2 \text{ and } E = \hbar\omega \text{ !}$$

Compare to plane wave at $(x = 5, t = 1)$



Energy and momentum

Energy and **momentum** are as expected

1. Amplitude varies in space as e^{ikx}, so particle has momentum $\hbar k$

2. Amplitude varies in time as $e^{-i\omega t}$, so particle has energy $\hbar\omega$