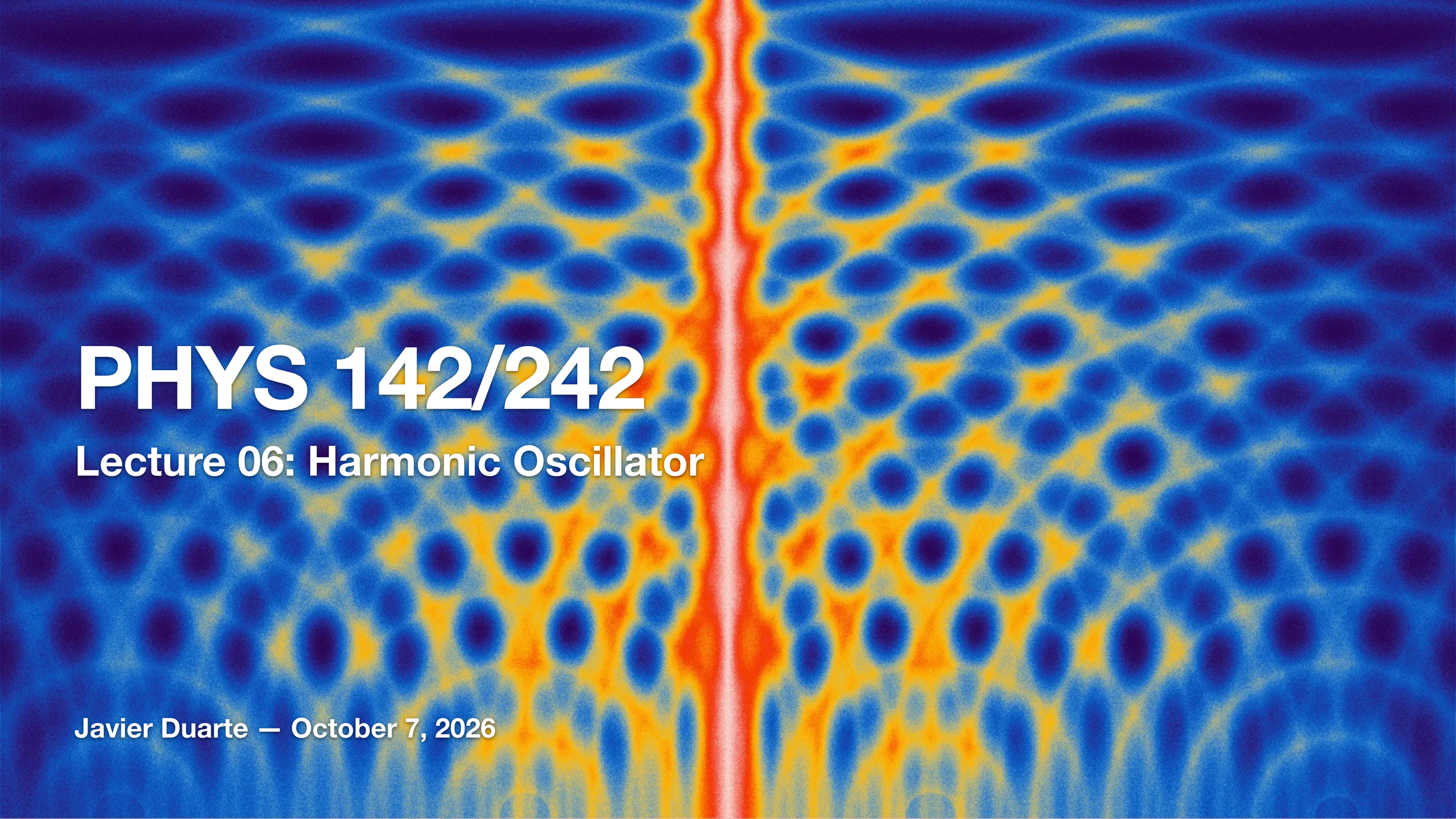
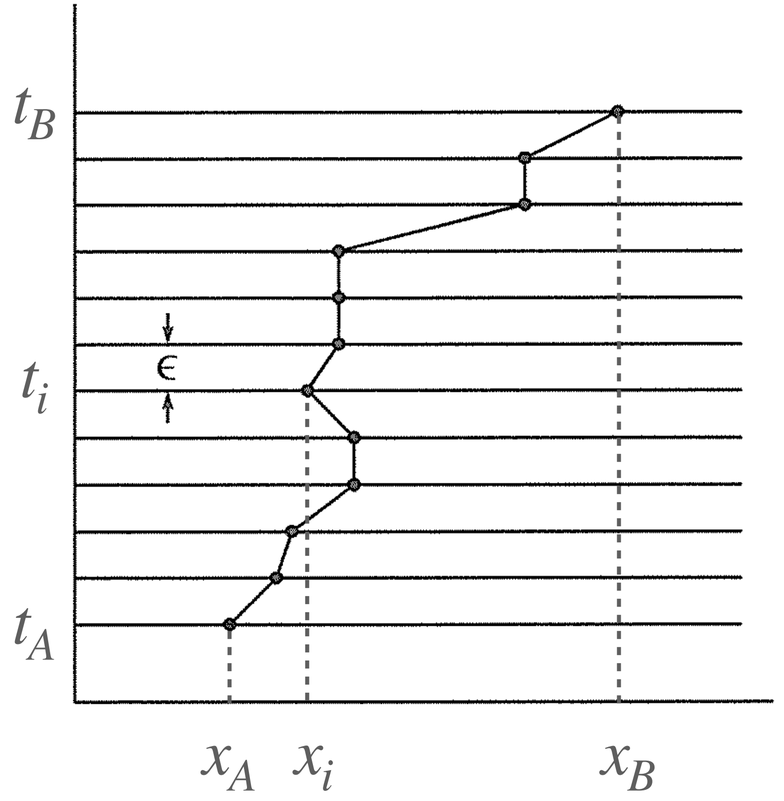


PHYS 142/242

Lecture 06: Harmonic Oscillator

Javier Duarte — October 7, 2026

Path integral



The path integral is

$$K(x_B, t_B; x_A, t_A) = \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S[B,A]}$$

$$= \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar}S[B,A]}$$

Path integral for a free particle

Last time: solved the path integral for a free particle

$$K(x_B, t_B; x_A, t_A) = \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i \hbar \epsilon}{m} \right)^{-\frac{N}{2}} \exp \left[\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2 \right]$$

$$= \left(\frac{m}{2\pi i \hbar (t_B - t_A)} \right)^{1/2} \exp \left[\frac{im(x_B - x_A)^2}{2\hbar(t_B - t_A)} \right]$$

Examined its behavior and found it to be consistent with a particle moving with **momentum** $m(x_B - x_A)/(t_B - t_A)$ and **energy** $\frac{1}{2} m((x_B - x_A)/(t_B - t_A))^2$

Wave functions and Schrödinger Equation

The propagator can tell us the probability amplitude to go from one state (described by a wave function) to another

$$\psi(x', t') = \int_{-\infty}^{\infty} K(x', t'; x, t)\psi_0(x, t)dx$$

The free propagator satisfies the Schrödinger equation!

$$-\frac{\hbar}{i}\frac{\partial K(x_B, t_B; x_A, t_B)}{\partial t_B} = -\frac{\hbar^2}{2m}\frac{\partial^2 K(x_B, t_B; x_A, t_A)}{\partial x_B^2} \text{ for } t_B > t_A$$

This implies that the free particle wave function also satisfies the Schrödinger equation

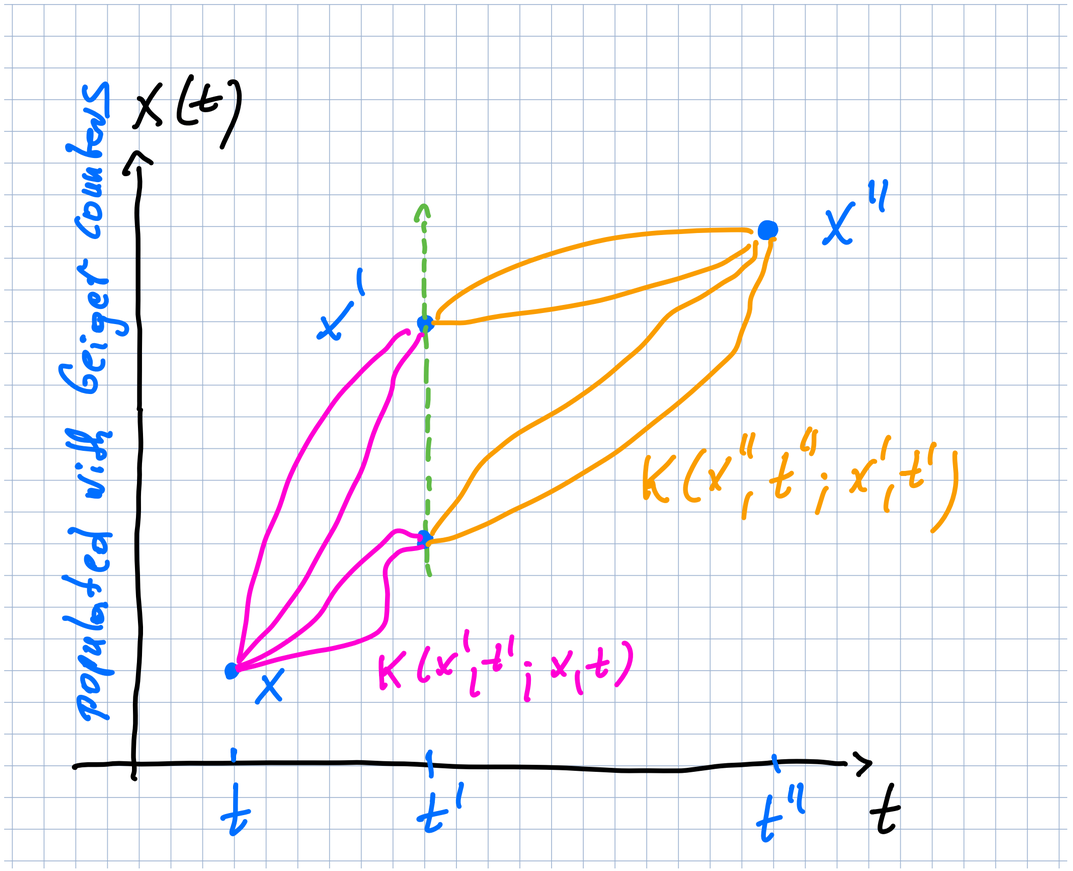
$$-\frac{\hbar}{i}\frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x, t)}{\partial x^2}$$

Chaining Propagators

$$\psi(x', t') = \int_{-\infty}^{\infty} K(x', t'; x, t)\psi_0(x, t)dx$$

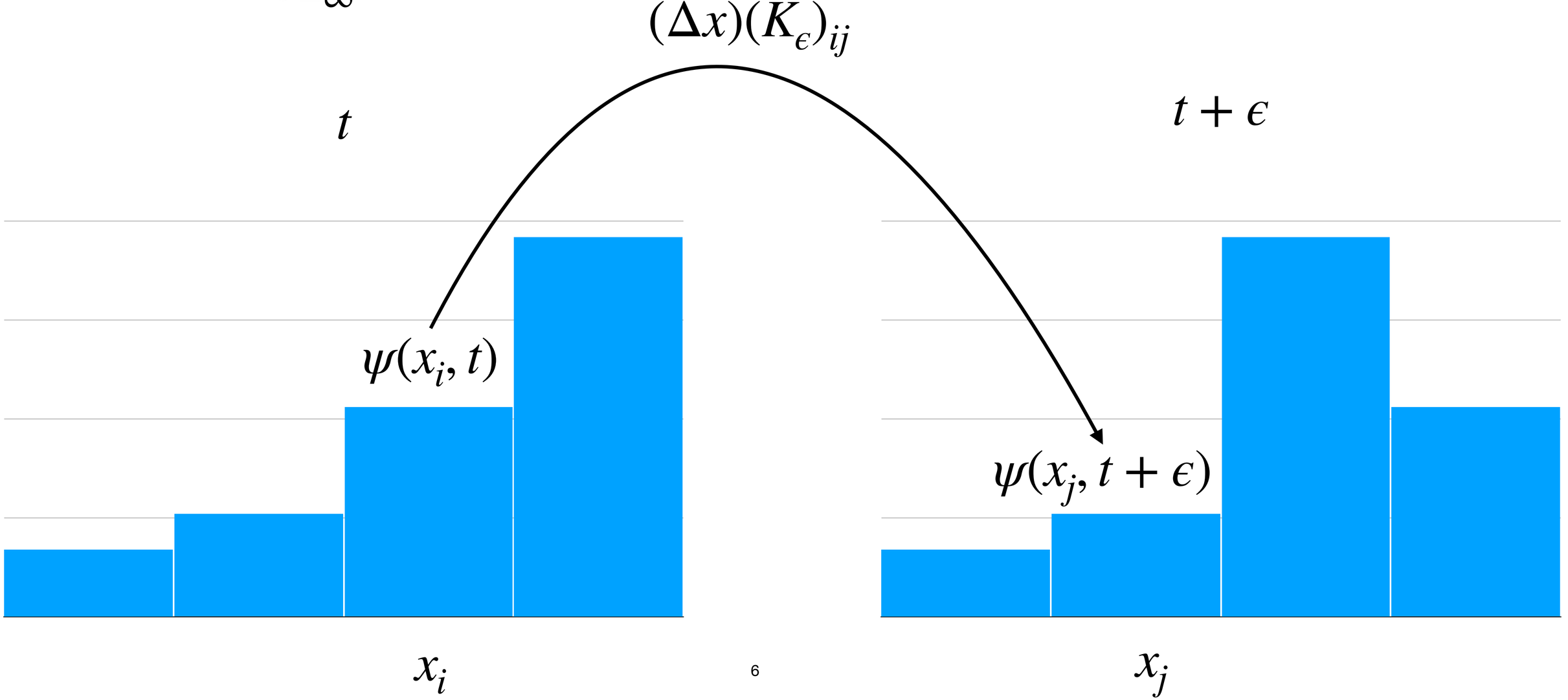
$$\widetilde{\psi}(x'', t'') = \int_{-\infty}^{\infty} K(x'', t''; x', t')\psi(x', t')dx'$$

$$\Rightarrow \widetilde{\psi}(x'', t'') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x'', t''; x', t')K(x', t'; x, t)\psi_0(x, t)dxdx'$$



Discretization

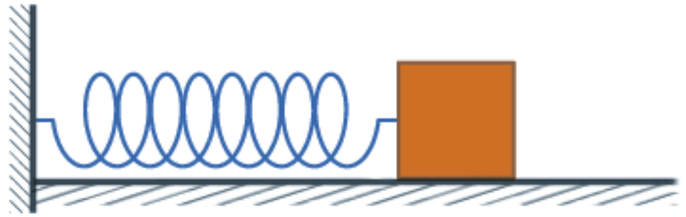
$$\psi(x', t+\epsilon) = \int_{-\infty}^{\infty} K(x', \epsilon; x, 0)\psi(x, t)dx \rightarrow \psi(x_j, t+\epsilon) = \Delta x K(x_j, \epsilon; x_i, 0)\psi(x_i, t)$$

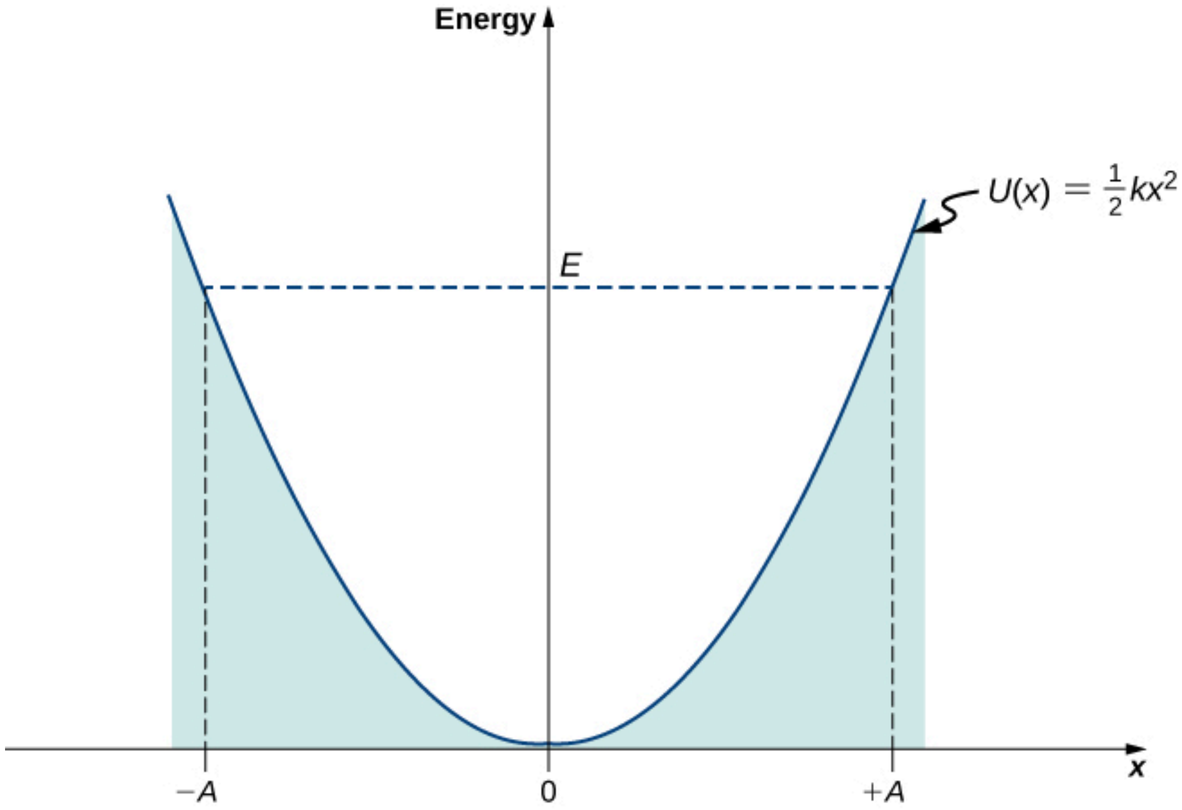


Harmonic Oscillator

- Reminder of the harmonic oscillator (quadratic potential)
  $L = K - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2$
- Classically, it models a block on a spring, or a (small-angle) pendulum, etc. For a spring, frequency is $\omega = \sqrt{k/m}$
- The motion is confined between turning points at $x = \pm\sqrt{C^2 + D^2}$.
- The total energy is $E = \frac{1}{2}k\sqrt{C^2 + D^2}$
- The general solution is

$$x(t) = C\sin\omega t + D\cos\omega t$$





Harmonic Oscillator Action

What is the classical action?

$$S = \int_{t_A}^{t_B} L dt = \int_{t_A}^{t_B} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \right) dt$$
$$= \frac{m\omega}{2 \sin \omega T} [(x_B^2 + x_A^2) \cos \omega T - 2 x_B x_A]$$

where $T = t_B - t_A$.

Main tricks are to (1) solve for C, D based on boundary conditions, (2) use $\ddot{x} = -\omega^2 x$ and integration by parts to solve for S, (3) Differentiate to get $\dot{x}(t)$ and (4) plug it all in

Harmonic Oscillator Action

(1) For simplicity, let's use $t_A = 0,\ t_B = T$. So $x(0) = x_A,\ x(T) = x_B$ gives

$$C = \frac{x_B - x_A \cos \omega T}{\sin \omega T},\ D = x_A$$

(2) Using integration by parts and $\ddot{x} = -\omega^2 x$

$$\int_{t_A}^{t_B} \dot{x}^2 = \int_{t_A}^{t_B} \left(\frac{d}{dt}(x\dot{x}) - x\ddot{x} \right) dt = x\dot{x}\Big|_{t_A}^{t_B} - \int_{t_A}^{t_B} x\ddot{x}dt = x\dot{x}\Big|_{t_A}^{t_B} + \omega^2 \int_{t_A}^{t_B} x^2 dt$$

Plugging this back into S,

$$S = \frac{1}{2} m\, x\dot{x}\Big|_{t_A}^{t_B} + \frac{1}{2} m\omega^2 \int_{t_A}^{t_B} x^2 dt - \frac{1}{2} m\omega^2 \int_{t_A}^{t_B} x^2 dt = \frac{1}{2} m\, x\dot{x}\Big|_{t_A}^{t_B}$$

Harmonic Oscillator Action

(3) So we need to differentiate and plug in the C, D we solved earlier

$$\dot{x}(t) = \omega C \cos\omega t - \omega D \sin\omega t = \omega \left(\frac{x_B - x_A \cos\omega T}{\sin\omega T} \right) \cos\omega t - \omega x_A \sin\omega t$$

(4) Now we plug it all in and do some algebra

$$S = \frac{1}{2} m\, x\dot{x} \Big|_{t_A}^{t_B} = \frac{1}{2} m \left[x_B \dot{x}(T) - x_A \dot{x}(0) \right]$$

$$= \frac{m\omega}{2 \sin\omega T} [(x_B^2 + x_A^2)\cos\omega T - 2 x_B x_A]$$

Example: Quantum harmonic oscillator

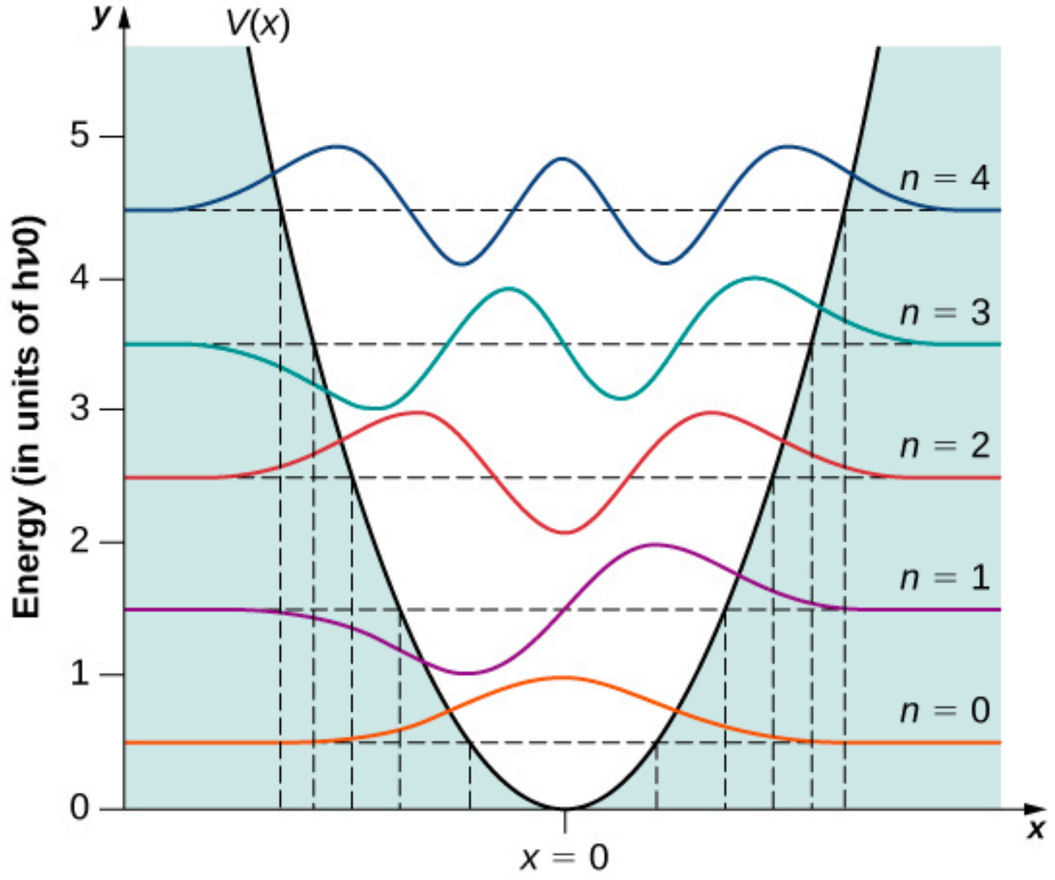
- Time-independent Schrödinger equation:

$$-\frac{\hbar}{2m}\frac{d^2\psi(x)}{dx^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\psi(x)$$

- **Ladder** of allowed energy levels

$$E_n=(n+\tfrac{1}{2})\hbar\omega \text{ with } n=0,1,2,3,\ldots$$

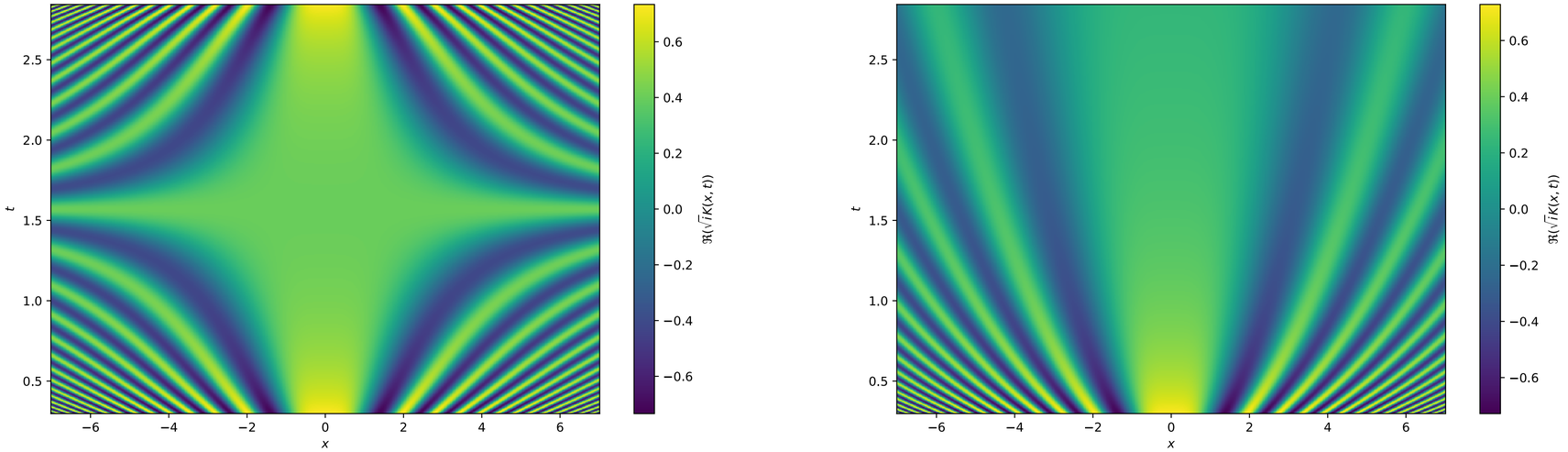
- Wave functions for each energy level
  $\psi_n(x)=N_ne^{-\beta^2x^2/2}H_n(\beta x)$
  - Note $H_0(y)=1$ so ground state is a Gaussian



Propagator for Harmonic Oscillator

- Propagator is given by

$$K(x_b, t_b; x_a, t_a) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T} \right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin \omega T} [(x_a^2 + x_b^2) \cos \omega T - 2x_a x_b] \right)$$



- Note, like the free particle propagator, it is of the form $K(B, A) = C(T)\exp[iS_{\text{cl}}(B, A)/\hbar]$
- Can we understand why?

Separable Lagrangians

- Let's break up the Lagrangian by considering any path to be a deviation y from the classical trajectory x_{cl}

$$L(x_{\text{cl}} + y, \dot{x}_{\text{cl}} + \dot{y}) = \frac{1}{2} m(\dot{x}_{\text{cl}} + \dot{y})^2 - \frac{1}{2} m\omega^2 (x_{\text{cl}} + y)^2$$

- $$= \frac{1}{2}\left(m\dot{x}_{\text{cl}}^2 + m\omega^2 x_{\text{cl}}^2\right) - (m\dot{x}_{\text{cl}}\dot{y} + m\omega^2 x_{\text{cl}} y) + \frac{1}{2}\left(m\dot{y}^2 + m\omega^2 y^2\right)$$

- The first term is the action for the classical trajectory
- The second term won't contribute to the action by the Euler-Lagrange equations! This is because when we integrate this over time, we can turn this into $\left(-m\ddot{x} + m\omega^2 x\right) y$ and $\ddot{x} = -\omega^2 x$
- The third term is the action for the deviation

Separable Lagrangians

- The action is separable because the propagator has the form:
- $$K(B,A) = C(T)\exp\left[\frac{i}{\hbar}S_{\text{cl}}(B,A)\right]\int_{\text{all paths}}\exp\left[\frac{i}{\hbar}\int_{t_A}^{t_B}\left(\frac{1}{2}m\dot{y}^2+\frac{1}{2}m\omega^2 y^2\right)dt\right]$$
- This integral over all the paths depends only on the elapsed time $T = t_b - t_a$ and not on x_A or x_B
- Shankar argues this is because every deviation has no knowledge of the endpoints of the trajectory since $y = \dot{y} = 0$ there
- Therefore, we can absorb this function into the normalization factor $C(T)$!
- This turns out to work for any potential that is quadratic in x and $\dot{x}$, so we expect the propagator to be of the form $K(B,A) = C(T)\exp[iS_{\text{cl}}(B,A)/\hbar]$

Hint for Assignment 1

In the assignment, the mean kinetic energy can be evaluated as a double integral

$$\langle K \rangle = \int\int dx dx' \frac{1}{2} m \left(\frac{x' - x}{\epsilon} \right)^2 \left| K(x', \epsilon; x, 0) \psi(x, t) \right|^2$$

However, there is another way, by integrating by parts

$$\langle K \rangle = \int \bar{\psi}(x) \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx = \frac{\hbar^2}{2} \int \left| \frac{\partial \psi}{\partial x} \right|^2 dx$$

which reduces it to just one integral