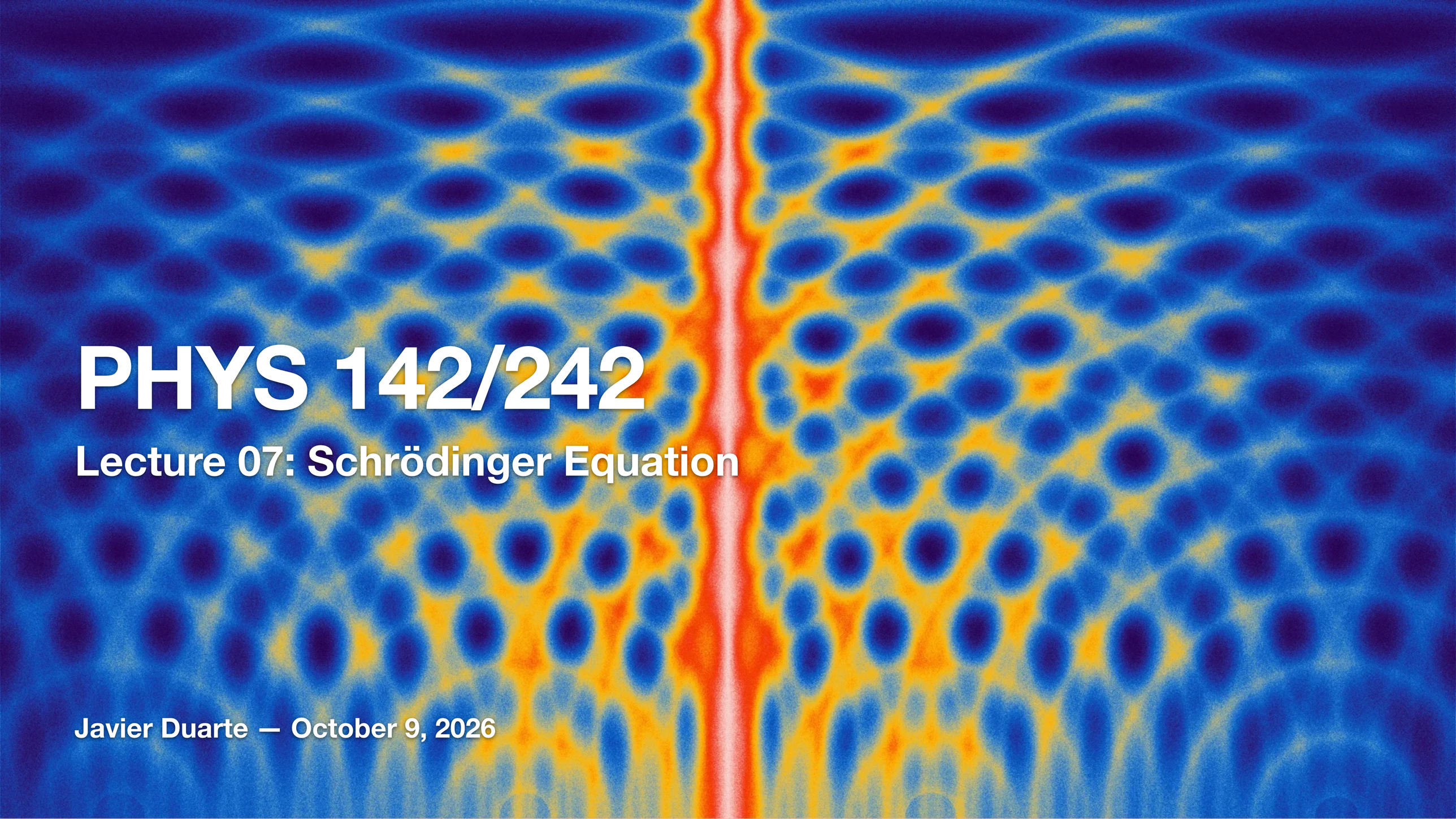


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Lecture 07: Schrödinger Equation

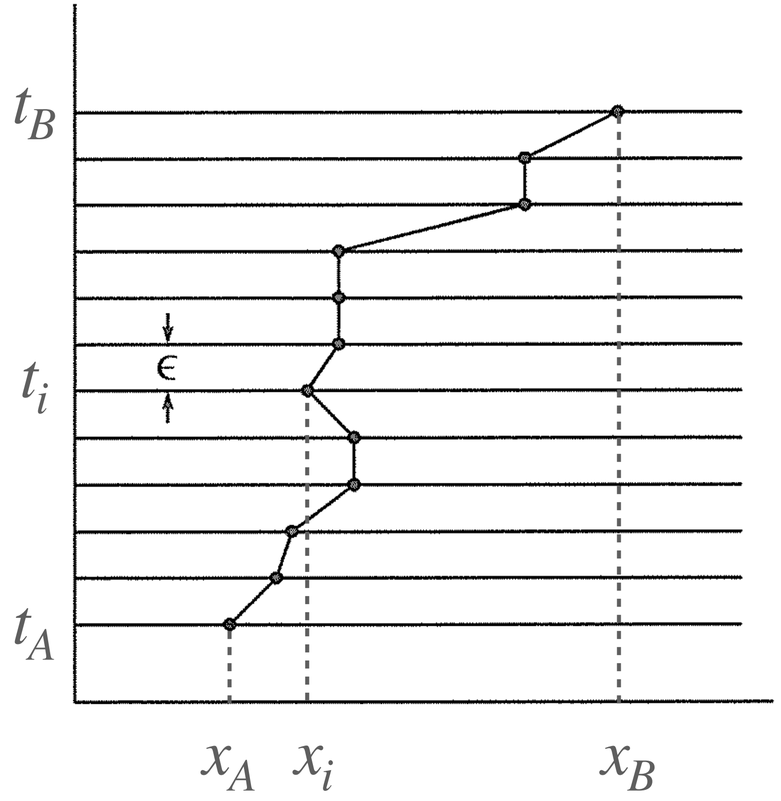
Javier Duarte — October 9, 2026

Path integral

The path integral is

$$K(x_B, t_B; x_A, t_A) = \int \mathscr{D}x(t) e^{\frac{i}{\hbar}S[B,A]}$$

$$= \lim_{\epsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \left(\frac{2\pi i\hbar\epsilon}{m} \right)^{-\frac{N}{2}} e^{\frac{i}{\hbar}S[B,A]}$$



Propagator tells us the probability amplitude to go from state $\psi(x, t)$ to $\psi(x', t')$

$$\psi(x', t') = \int_{-\infty}^{\infty} K(x', t'; x, t)\psi(x, t)dx$$

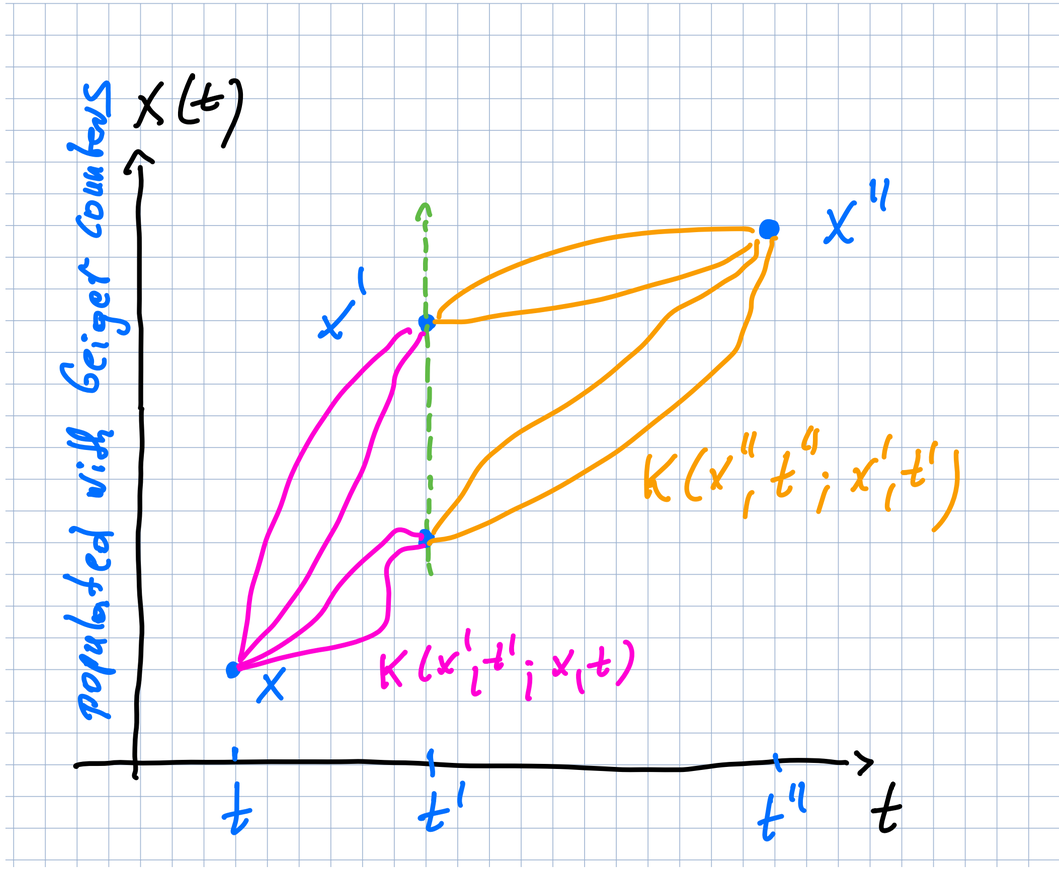
Chaining propagators

$$\psi(x',t') = \int_{-\infty}^{\infty} K(x',t';x,t)\psi(x,t)dx$$

$$\psi(x'',t'') = \int_{-\infty}^{\infty} K(x'',t'';x,t)\psi(x,t)dx$$

$$\psi(x'',t'') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x'',t'';x',t')K(x',t';x,t)\psi(x,t)dx'dx$$

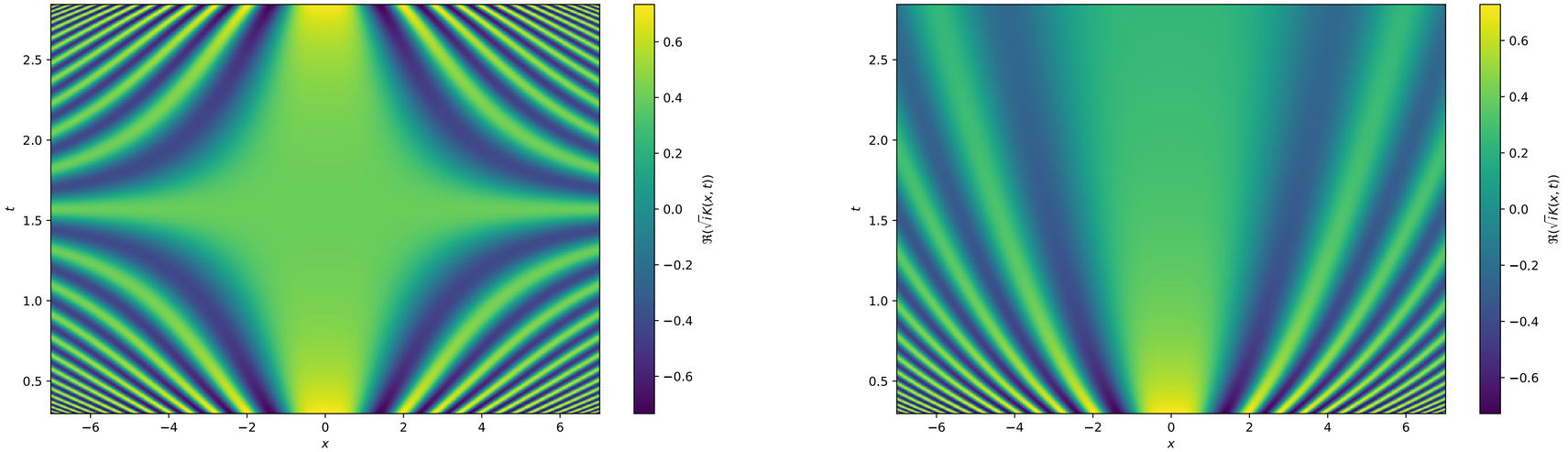
$$\Rightarrow K(x'',t'';x,t) = \int_{-\infty}^{\infty} K(x'',t'';x',t')K(x',t';x,t)dx'$$



Propagator for Harmonic Oscillator

- Propagator is given by

$$K(x_b, t_b; x_a, t_a) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T} \right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin \omega T} [(x_a^2 + x_b^2) \cos \omega T - 2x_a x_b] \right)$$



- Note, like the free particle propagator, it is of the form $K(B, A) = C(T)\exp[iS_{\text{cl}}(B, A)/\hbar]$
- Can we understand why?

Separable Lagrangians

- Let's break up the Lagrangian by considering any path to be a deviation y from the classical trajectory x_{cl}

$$L(x_{\text{cl}} + y, \dot{x}_{\text{cl}} + \dot{y}) = \frac{1}{2} m(\dot{x}_{\text{cl}} + \dot{y})^2 - \frac{1}{2} m\omega^2 (x_{\text{cl}} + y)^2$$

- $$= \frac{1}{2}\left(m\dot{x}_{\text{cl}}^2 + m\omega^2 x_{\text{cl}}^2\right) - (m\dot{x}_{\text{cl}}\dot{y} + m\omega^2 x_{\text{cl}} y) + \frac{1}{2}\left(m\dot{y}^2 + m\omega^2 y^2\right)$$

- The first term is the action for the classical trajectory
- The second term won't contribute to the action by the Euler-Lagrange equations! This is because when we integrate this over time, we can turn this into $\left(-m\ddot{x} + m\omega^2 x\right) y$ and $\ddot{x} = -\omega^2 x$
- The third term is the action for the deviation

Separable Lagrangians

- The action is separable because the propagator has the form:
- $$K(B,A) = C(T)\exp\left[\frac{i}{\hbar}S_{\text{cl}}(B,A)\right]\int_{\text{all paths}}\exp\left[\frac{i}{\hbar}\int_{t_A}^{t_B}\left(\frac{1}{2}m\dot{y}^2+\frac{1}{2}m\omega^2y^2\right)dt\right]$$
- This integral over all the paths depends only on the elapsed time $T = t_b - t_a$ and not on x_A or x_B
- Shankar argues this is because every deviation has no knowledge of the endpoints of the trajectory since $y = \dot{y} = 0$ there
- Therefore, we can absorb this function into the normalization factor $C(T)$!
- This turns out to work for any potential that is quadratic in x and $\dot{x}$, so we expect the propagator to be of the form $K(B,A) = C(T)\exp[iS_{\text{cl}}(B,A)/\hbar]$

Hint for Assignment 1

In the assignment, the mean kinetic energy can be evaluated as a double integral

$$\langle K \rangle = \int\int dx dx' \frac{1}{2} m \left(\frac{x' - x}{\epsilon} \right)^2 \left| K(x', \epsilon; x, 0) \psi(x, t) \right|^2$$

However, there is another way, by integrating by parts

$$\langle K \rangle = \int \bar{\psi}(x) \left(-\frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} \right) \psi(x) dx = \frac{\hbar^2}{2} \int \left| \frac{\partial \psi}{\partial x} \right|^2 dx$$

which reduces it to just one integral

Path integral examples

Solved path integral for a free particle

$$K_{\text{free}}(x_b, t_b; x_a, t_a) = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar T}\right]$$

and for the harmonic oscillator

$$K_{\text{HO}}(x_b, t_b; x_a, t_a) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega T}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar \sin\omega T}[(x_a^2 + x_b^2)\cos\omega T - 2x_a x_b]\right)$$

Both are of the form

$$K(x_b, t_b; x_a, t_a) = C(T)\exp\left[\frac{i}{\hbar}S_{\text{cl}}(b, a)\right]$$

This form is correct for all Lagrangians that are quadratic forms (depend only on x^2 and $\dot{x}^2$)

Harmonic oscillator: small angle approx.

Note that for $\omega T \ll 1$, $\sin \omega T \sim \omega T, \cos \omega T \sim 1$ so the path integral for the harmonic oscillator reduces to

$$K_{\mathrm{HO}}(x_b, t_b; x_a, t_a) = \left(\frac{m\omega}{2\pi i\hbar\omega T} \right)^{1/2} \exp\left(\frac{im\omega}{2\hbar\omega T}[(x_a^2 + x_b^2) - 2x_a x_b] \right)$$

$$= \left(\frac{m}{2\pi i\hbar T} \right)^{1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar T} \right]$$

$$= K_{\mathrm{free}}(x_b, t_b; x_a, t_a)$$

Why?

If we move step forward by a time step T that is very small compared with the period of the oscillator $2\pi/\omega$, the particle appears to move as if it is free

Schrödinger Equation

Note: the free propagator satisfies the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial K(x_B,t_B;x_A,t_B)}{\partial t_B} = -\frac{\hbar^2}{2m}\frac{\partial^2 K(x_B,t_B;x_A,t_A)}{\partial x_B^2} \text{ for } t_B > t_A$$

and the free particle wave function also satisfies the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2}$$

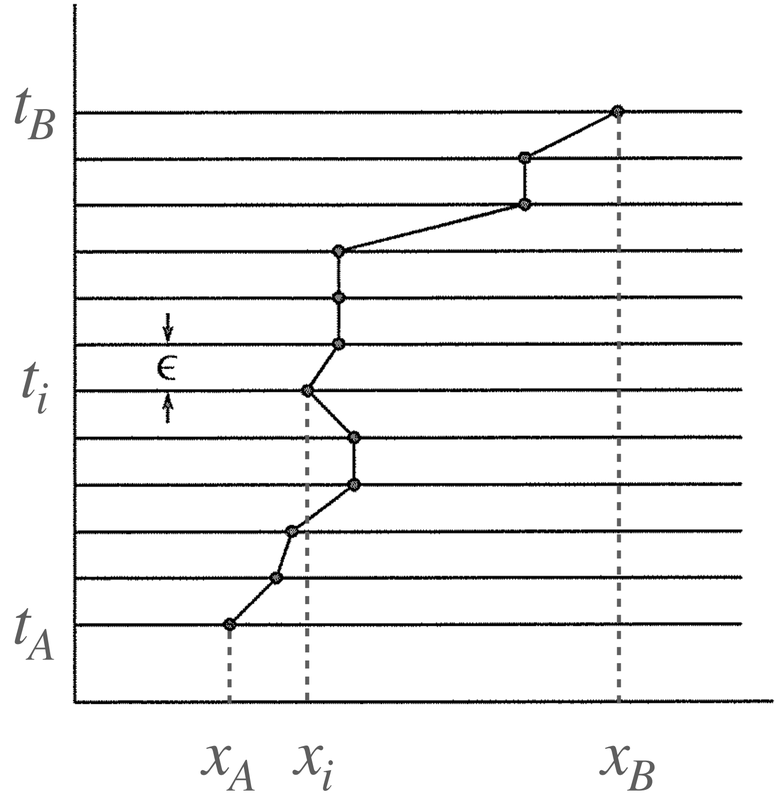
What if there is a potential $V(x,t)$?

Our goal today is to “derive” the Schrödinger equation from the path integral

Action in a potential

The action with a potential is

$$S[B,A] = \int_{t_A}^{t_B} \left(\frac{1}{2} m\dot{x}^2 - V(x,t) \right) dt$$



Let's return to our "zig-zag" paths. The action can be written as

$$S[B,A] = \sum_{i=1}^{N} S[i,i-1] = \sum_{i=1}^{N} \frac{m}{2} \frac{(x_i - x_{i-1})^2}{\epsilon} - \epsilon V\left(\frac{x_{i-1} + x_i}{2}, t_i \right)$$

Deriving Schrödinger (1)

We will approximate the time derivative of ψ

$$\psi(x, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m(x-y)^2}{2\epsilon}\right] \exp\left[-\frac{i}{\hbar}\epsilon V\left(\frac{x+y}{2}, t\right)\right] \psi(y,t)dy,$$

where $C = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}$.

First term oscillates rapidly for large $x-y$.

Substitute $y = x+\eta$ and expect contribution only for small η

$$\psi(x, t+\epsilon) = C\int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right] \exp\left[-\frac{i\epsilon}{\hbar}V\left(x+\frac{\eta}{2}, t\right)\right] \psi(x+\eta,t)d\eta\,.$$

Integral contributes if $0 \leq |\eta| \leq \sqrt{\frac{\hbar\epsilon}{m}}$ (stationary phase approximation)

Deriving Schrödinger (2)

The left hand side can be expanded as a power series in ϵ and the right hand side as a power series in η

$$\psi(x,t) + \epsilon\frac{\partial\psi}{\partial t} = C\int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right]\left[1 - \frac{i\epsilon}{\hbar}V(x,t)\right]\left[\psi(x,t) + \eta\frac{\partial\psi}{\partial x} + \frac{1}{2}\eta^2\frac{\partial^2\psi}{\partial x^2}\right]d\eta$$

So we need to be able to evaluate Gaussian integrals of the form

$$C\int_{-\infty}^{\infty} \exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right]\eta^m d\eta, \qquad m = 0,1,2$$

More Gaussian integrals (1)

Now on the right hand side we have 3 kinds of Gaussian integrals.

The first we've discussed

$$\int_{-\infty}^{\infty} e^{iax^2} dx = \sqrt{\frac{\pi}{ia}}$$

The second is zero because it's an odd function $f(-x) = -f(x)$ and we're integrating from $-\infty$ to ∞

$$\int_{-\infty}^{\infty} e^{iax^2} x dx = 0$$

More Gaussian integrals (2)

The third can be solved with an interesting trick usually credited to Feynman.

The usual idea is to introduce an artificial parameter then take a derivate of this parameter to see if you can find an integrand that is easier to do.

We can also do this in reverse. Namely, we note that

$$\int_{-\infty}^{\infty} e^{iax^2} x^2 dx = \frac{1}{i}\frac{d}{da}\int_{-\infty}^{\infty} e^{iax^2} dx$$

$$= \frac{1}{i}\frac{d}{da}\sqrt{\frac{\pi}{ia}} = \frac{1}{2}\sqrt{\frac{i\pi}{a^3}}\,.$$

and since we already know the integral on the right, we can just plug it in then take the derivative of a!

Deriving Schrödinger (3)

Therefore, the 3 integrals are

$$C\int_{-\infty}^{\infty}\exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right]d\eta = C\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2} = 1$$

$$C\int_{-\infty}^{\infty}\exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right]\eta d\eta = 0$$

$$C\int_{-\infty}^{\infty}\exp\left[\frac{im\eta^2}{2\hbar\epsilon}\right]\eta^2 d\eta = \frac{i\hbar\epsilon}{m}$$

And

$$\psi(x,t) + \epsilon\frac{\partial\psi}{\partial t} = \psi(x,t) - \frac{i\epsilon}{\hbar}V(x,t)\psi(x,t) - \frac{\hbar\epsilon}{2im}\frac{\partial^2\psi}{\partial x^2}$$

Deriving Schrödinger (4)

Or rearranging, we arrive at the Schrödinger equation!

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x,t)\psi(x,t)$$

Next time, discussion of Hamiltonian operator, conservation of probability (unitarity), stationary states (energy eigenfunctions), and connection to the propagator trace