PHYS 142/242

Lecture 09: Double Well and Tunneling

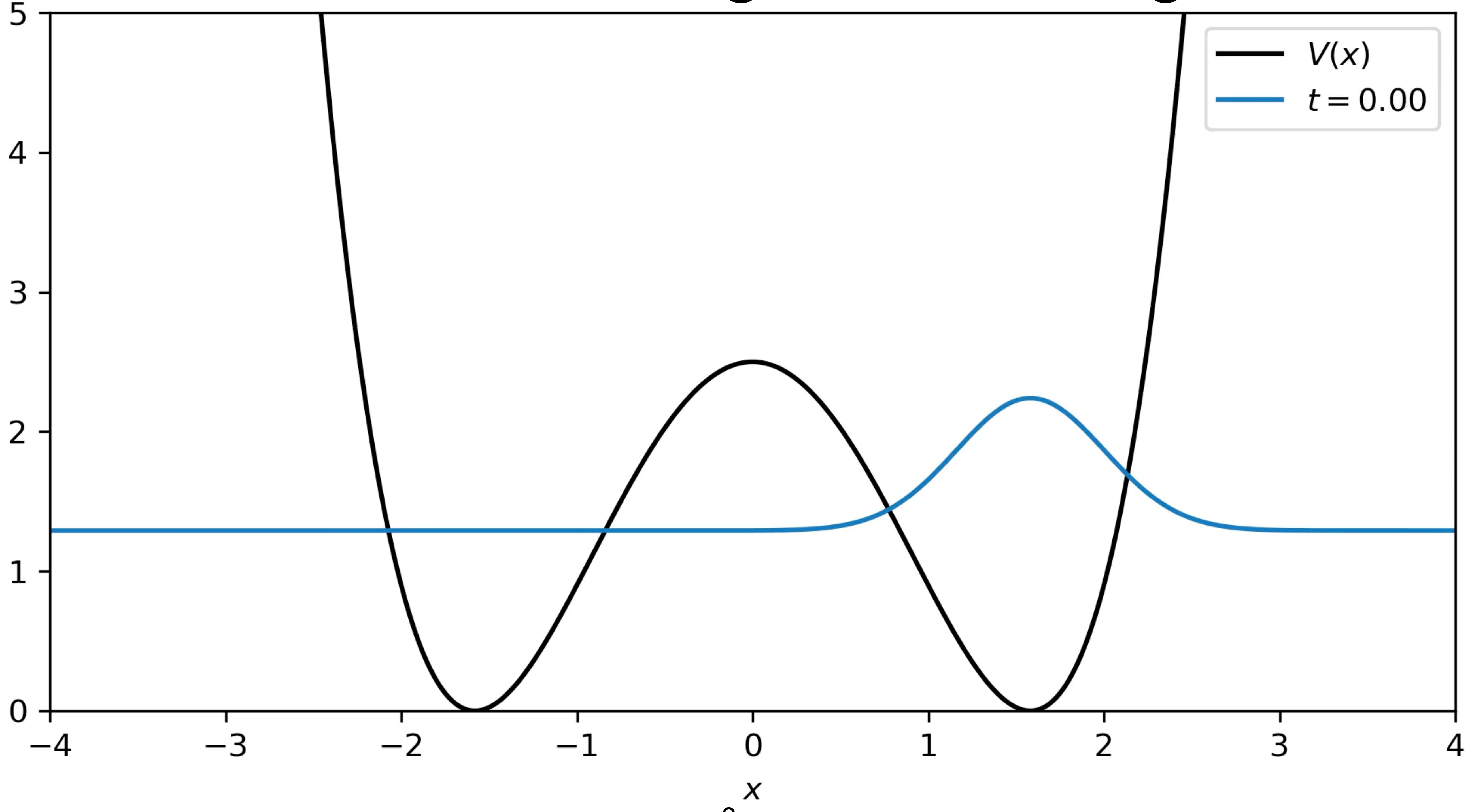
Javier Duarte — January 27, 2025

Homework reminder

- Half of grade will be from turning in first “draft”
  - Graded on effort and completeness (for all problems)
  - Solution released shortly afterward
- Half of grade will be from turning in corrected solution
  - Graded on effort and correctness (for all problems)
  - ***Note: DO NOT just turn in solutions; CORRECT your own first attempt***
- Report (pdf file) uploaded to Gradescope
- Code (zip file) uploaded to Gradescope
- Assignment 1 correction due Wednesday 8pm!

Quantum tunneling

- *Quantum tunneling* is the phenomenon in which a particle passes through a potential energy barrier that, according to classical mechanics, should not be passable because it doesn't have sufficient energy to surmount the barrier
- We will analyze this phenomena in Assignment 2 using the *double well potential*
- In particular, we are interested in finding the *tunneling time*



Double well potential

Consider the double well potential

$$V(x) = \alpha x^4 - 2x^2 + \frac{1}{\alpha}$$

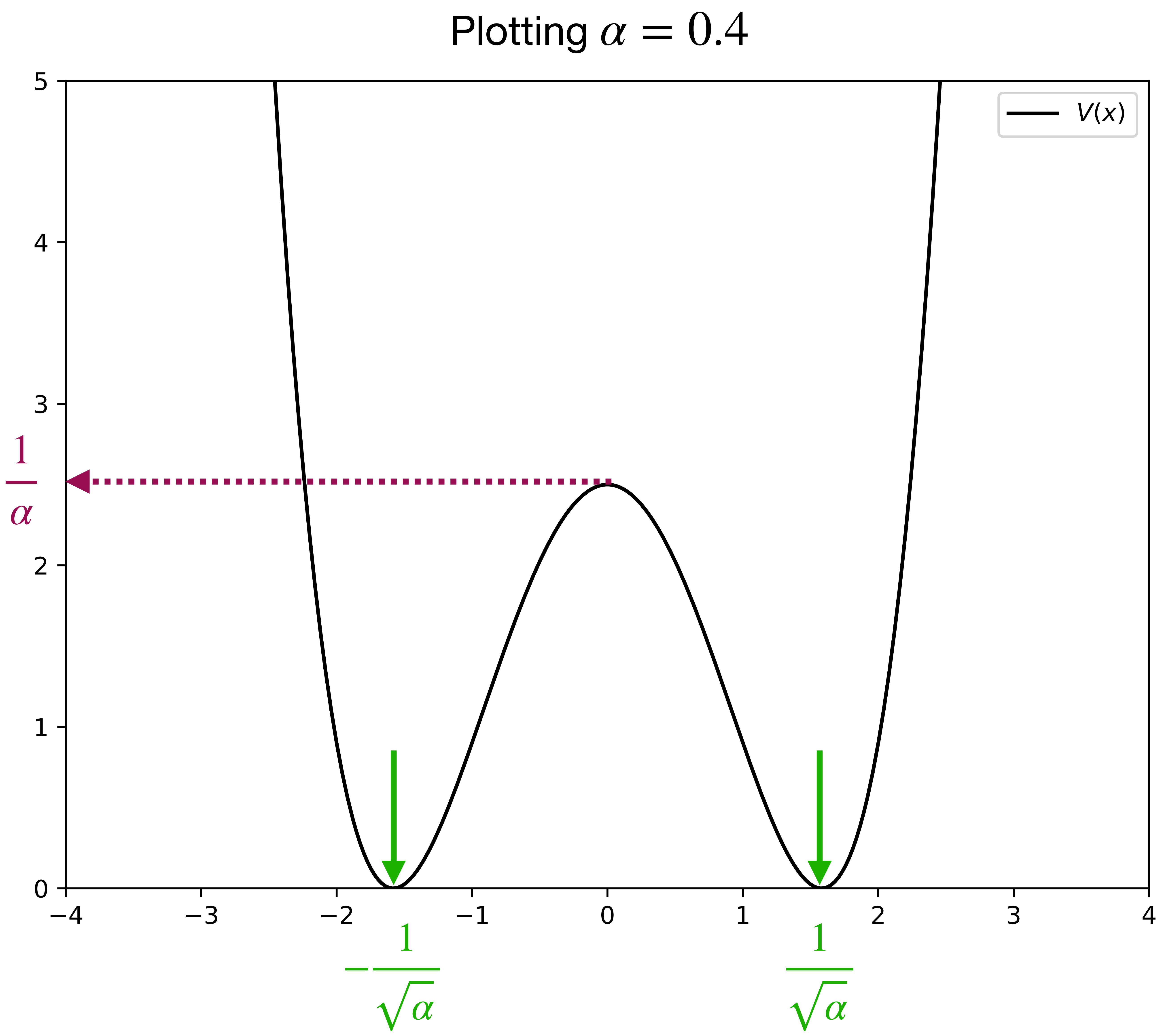
Two minima separated by a potential barrier

Barrier height: $V(0) = \frac{1}{\alpha}$

Minima:

$$V'(x_{\text{min}}) = 4\alpha x_{\text{min}}^3 - 4x_{\text{min}} = 0$$

$$\Rightarrow x_{\text{min}} = \pm\frac{1}{\sqrt{\alpha}}$$



Double well potential

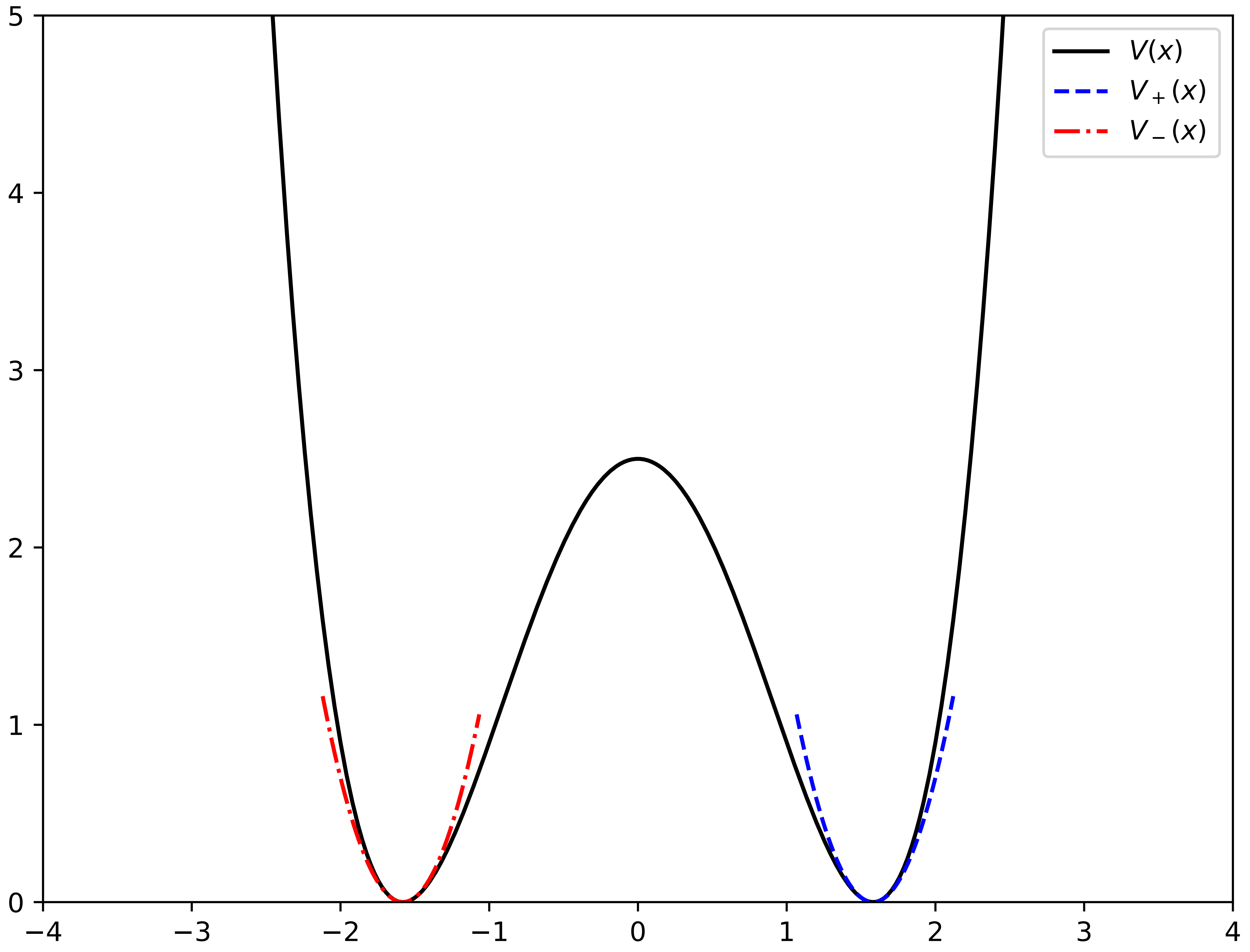
Near the minima, we approximate the potential using quadratic harmonic oscillator potentials

$$V''(x_{\text{min}}) = 12\alpha x_{\text{min}}^2 - 4$$
$$= 12\alpha\left(\frac{1}{\alpha}\right) - 4$$
$$= 8$$

$$V_+(x) = 4(x - x_{\text{min}})^2$$

$$V_-(x) = 4(x + x_{\text{min}})^2$$

Equivalent to $m = 1,\ \omega = 2\sqrt{2}$



Double well potential

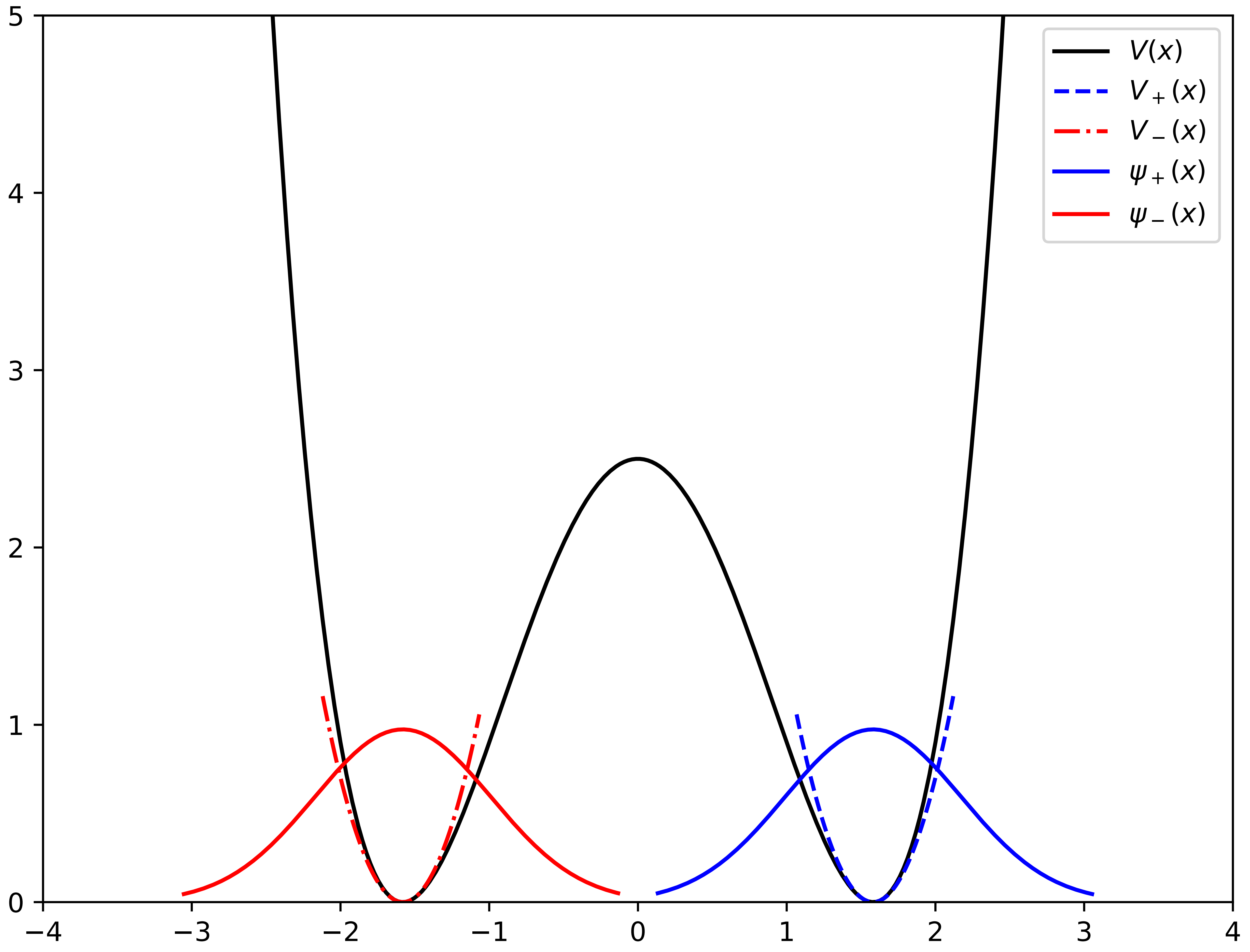
Typically, we solve the eigenvalue equation

$$-\frac{1}{2}\frac{\partial^2\psi_n(x)}{\partial x^2} + V(x)\psi_n(x) = E_n\psi_n(x)$$

But we already know the solutions for the HO

$$\psi_+(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x - x_{\min})^2\right)$$

$$\psi_-(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x + x_{\min})^2\right)$$



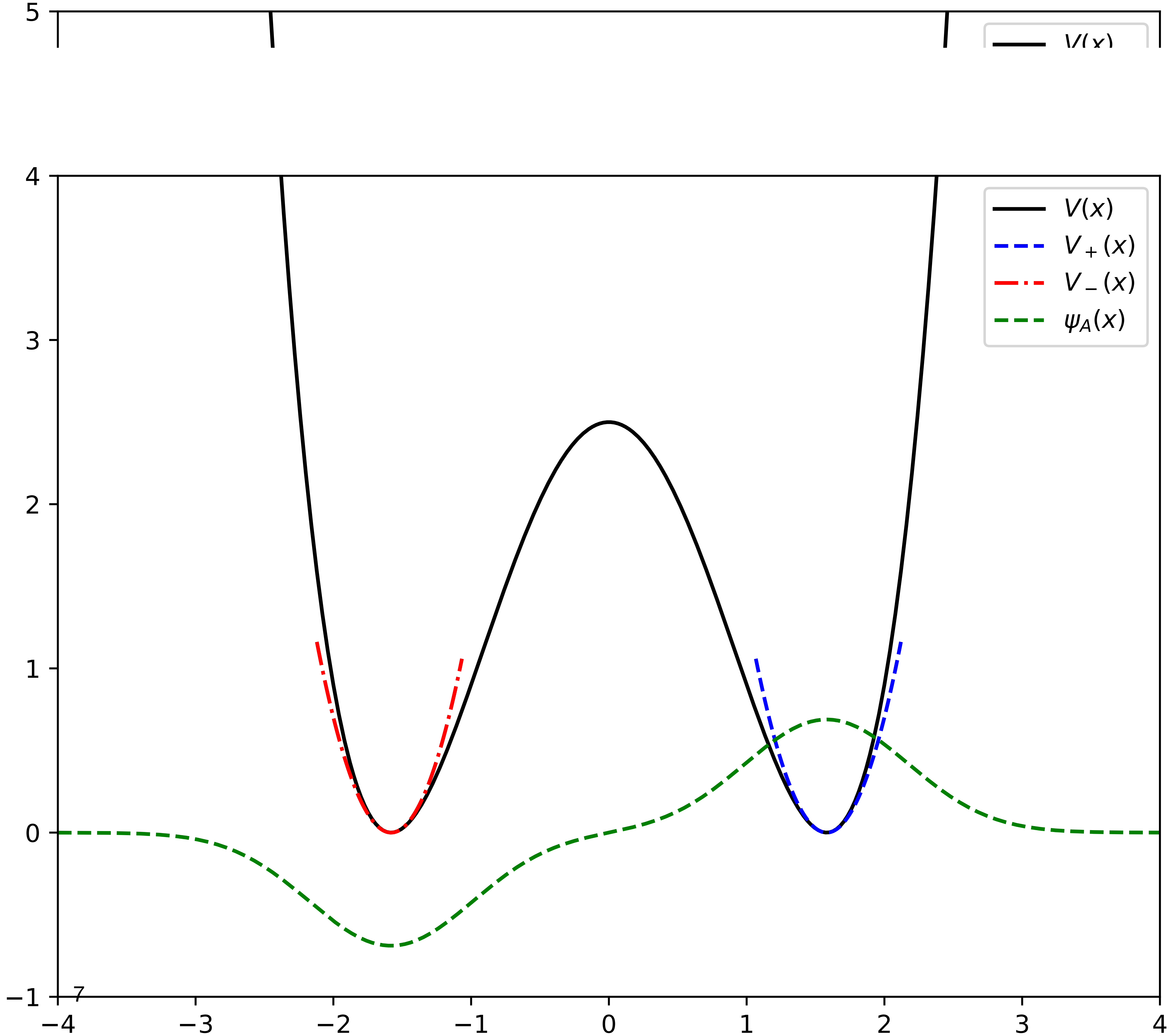
Double well potential

The ground state $\psi_0(x)$ and first excited state $\psi_1(x)$ of the full potential should be *symmetric* $\psi_0(x) = \psi_0(-x)$ and *antisymmetric* $\psi_1(x) = -\psi_1(-x)$, respectively

We can approximate them with

$$\psi_0(x) \approx \psi_S(x) = \frac{1}{\sqrt{2}}\left(\psi_+(x) + \psi_-(x)\right)$$

$$\psi_1(x) \approx \psi_A(x) = \frac{1}{\sqrt{2}}\left(\psi_+(x) - \psi_-(x)\right)$$

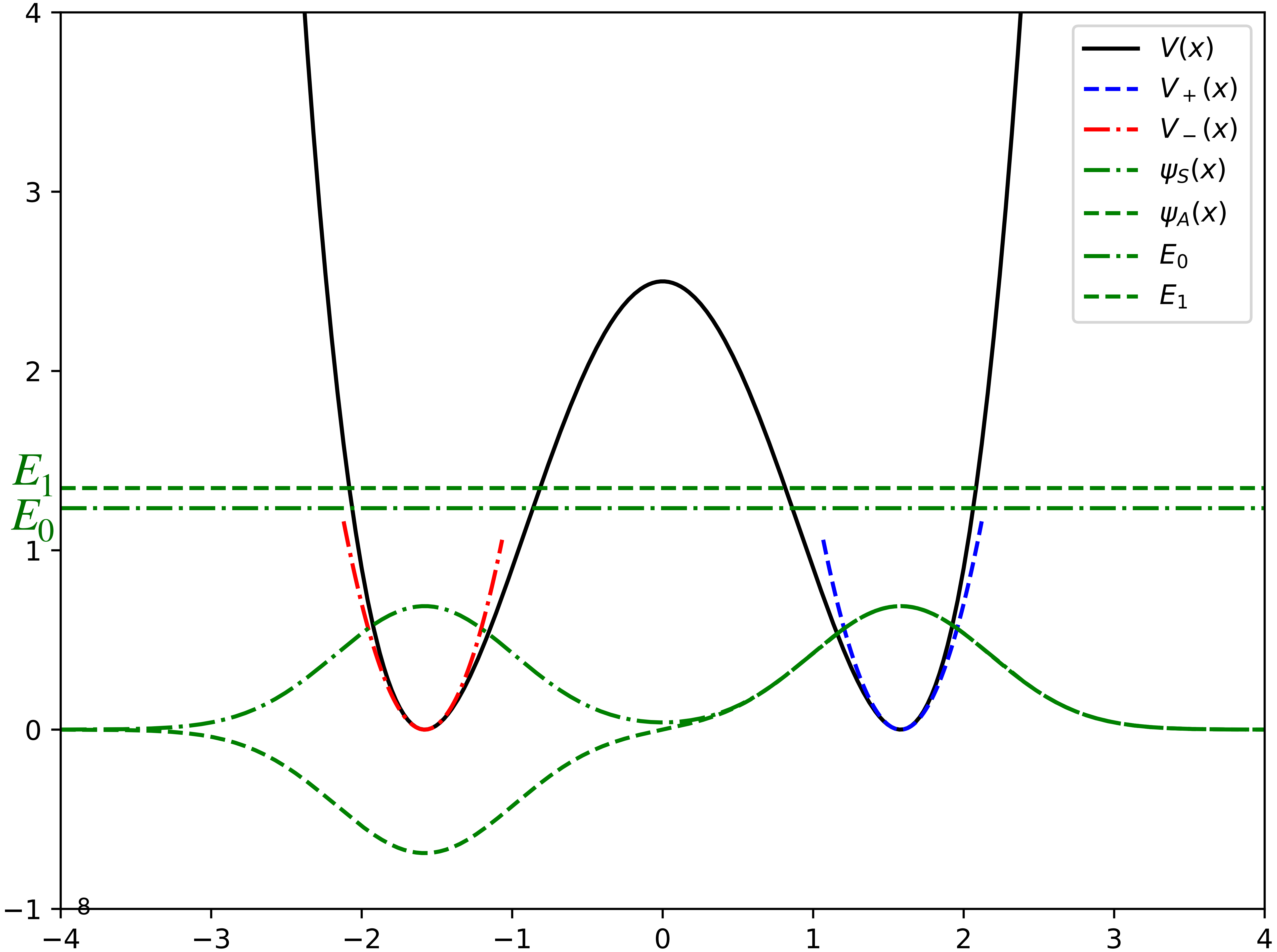


Double well potential

The two lowest energy eigenvalues
E_0 and E_1 are nearly degenerate
$\Delta E \equiv E_1 - E_0 \ll \frac{1}{2}(E_0 + E_1)$

If we know the energy eigenvalues, we can time evolve

$$\psi_S(x,t) = e^{-iE_0 t}\psi_S(x,0)$$
$$\psi_A(x,t) = e^{-iE_1 t}\psi_A(x,0)$$



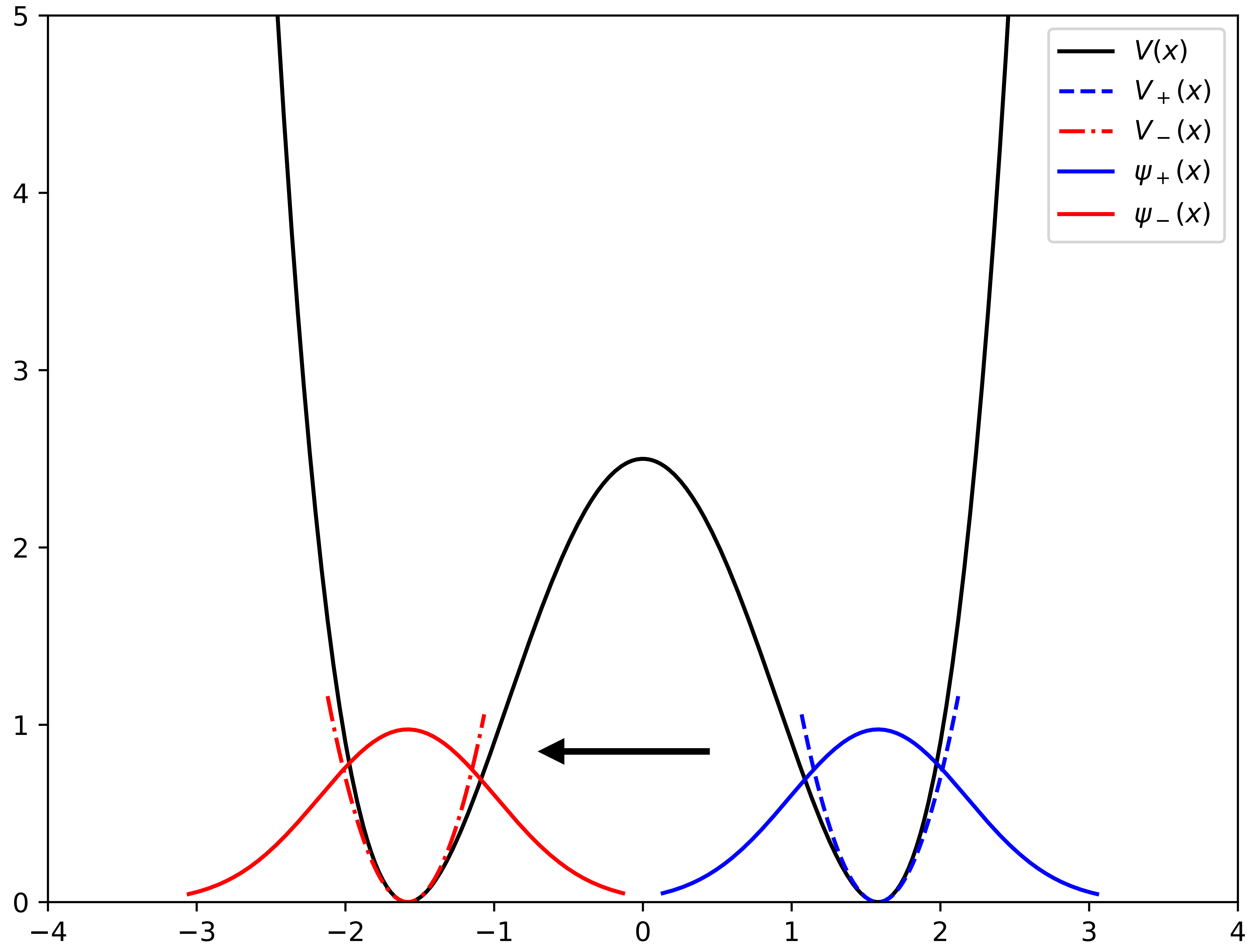
Tunneling time

Let's say we start in the right well

$$\psi(x,0) = \psi_+(x)$$

Define the tunneling time t_{tunnel} as the time it takes for the particle to fully tunnel into the left well

$$\psi(x, t_{\text{tunnel}}) \propto \psi_-(x)$$



Tunneling time

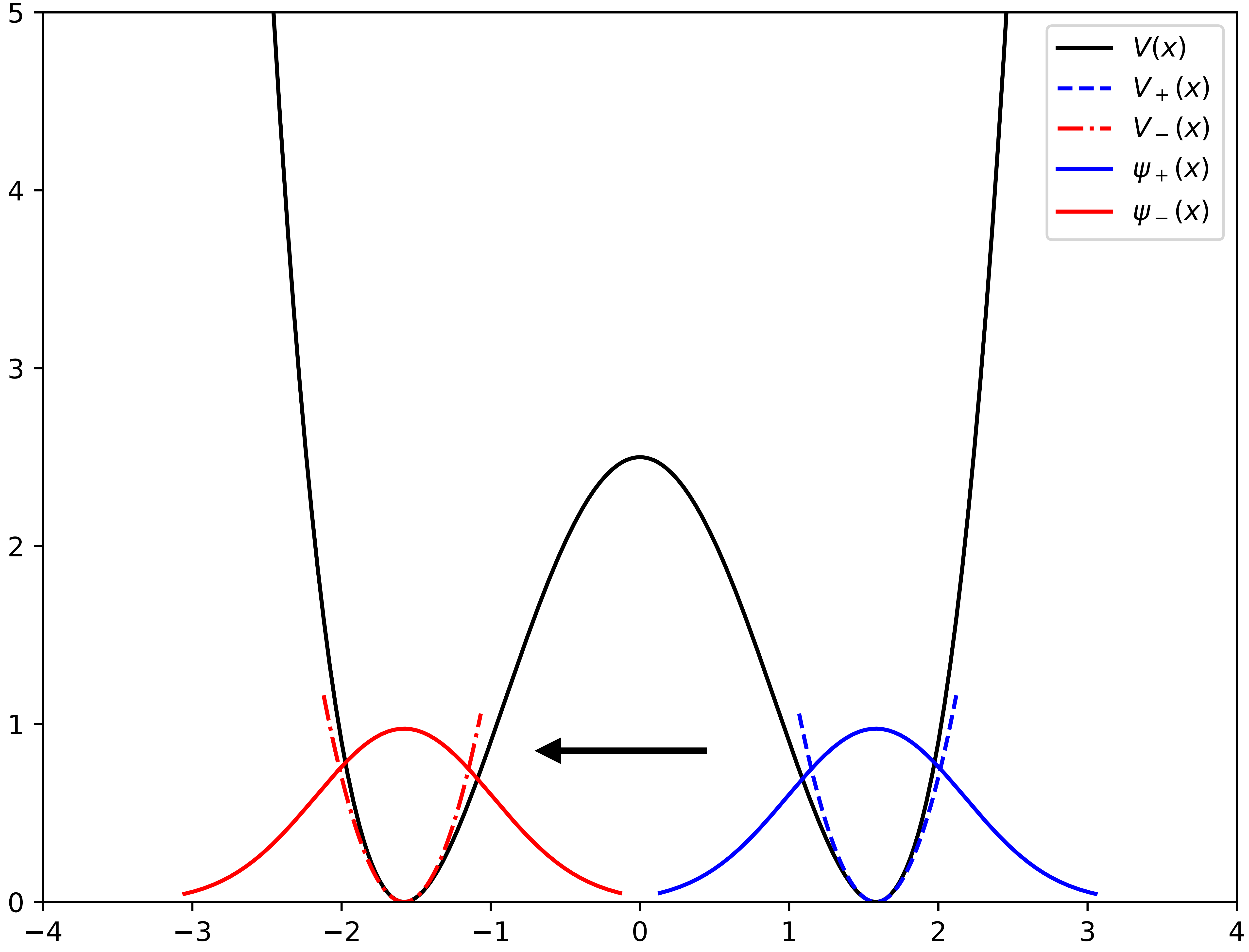
Note:
$\psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_S(x,0) + \psi_A(x,0)\right) = \psi_+(x)$

Time evolving it we get

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{-iE_0 t}\psi_S(x,0) + e^{-iE_1 t}\psi_A(x,0)\right)$$
$$= \frac{1}{\sqrt{2}}e^{-iE_0 t}\left(\psi_S(x,0) + e^{-i\Delta E t}\psi_A(x,0)\right)$$

We want
$\psi(x,t_{\text{tunnel}}) \propto \psi_-(x) = \frac{1}{\sqrt{2}}\left(\psi_S(x,0) - \psi_A(x,0)\right)$

So the sign has to flip
$e^{-i\Delta E t_{\text{tunnel}}} = -1$ or $t_{\text{tunnel}} = \pi/\Delta E$



Numerical results

K. Banerjee and S. P. Bhatnagar, "Two-well oscillator", Phys. Rev. D 18, 4767 (also uploaded to Canvas) numerically calculate the ground state and first excited state energy eigenvalues

Note: different convention $4\lambda = \alpha$ so $\lambda = 0.1$ corresponds to $\alpha = 0.4$

So for $\alpha = 0.4$, we have $E_0 = 1.2345$, $E_1 = 1.3469$, and $\Delta E = 0.1124$

$t_{\text{tunnel}} = \pi/\Delta E = 27.94$

TABLE I. Eigenvalues of the two-well oscillator in the small-λ regime. $\epsilon_n(\lambda)$ are the computed exact eigenvalues of the energy-shifted operator $H(1,\lambda) + (1/4\lambda)$, which is positive definite.

λ	ϵ_0 / ϵ_1	ϵ_2 / ϵ_3
0.01	1.404 048 605 297 7[a]	4.170 193 605 999 3
	1.404 048 605 297 7	4.170 193 605 999 3
0.02	1.393 527 585 044 2	4.092 028 112 820 5
	1.393 527 587 151 0	4.092 028 608 428 7
0.03	1.382 601 444 053 8	4.006 049 199 465 7
	1.382 605 783 831 4	4.006 655 466 749 5
0.04	1.371 122 236 557 5	3.901 359 951 813 1
	1.371 308 461 612 9	3.918 263 337 997 1
0.05	1.358 422 103 747 8	3.746 917 080 727 9
	1.360 133 597 773 3	3.848 838 300 057 4
0.07	1.323 374 074 208 5	3.342 216 720 258 7
	1.343 365 616 287 4	3.833 129 937 607 9
0.10	1.234 507 162 786 0	3.009 488 545 436 2
	1.346 940 868 922 5	4.043 546 039 767 6
0.15	1.062 499 247 956 5	3.033 667 276 570 6
	1.421 086 890 539 3	4.589 838 495 543 4
0.17	1.007 165 158 778 7	3.118 337 642 119 7
	1.464 225 132 421 2	4.816 923 221 196 9
0.20	0.941 750 342 076 9	3.270 377 801 715 3
	1.535 530 204 085 8	5.148 274 740 096 0

[a] Since near the minima the potentail function $\sim 2x^2 + O(\lambda^{1/2}x^3)$, $\epsilon_0 \to \sqrt{2}$ (ground-state energy in a potential $2x^2$) as $\lambda \to 0$. We find $\epsilon_0(\lambda = 0.001) = 1.413\,211\,965\,792$.

Quantum tunneling

- $t_{\text{tunnel}} = \pi / \Delta E = 27.94$
- Assignment 2 is to show this with Feynman path integral approach