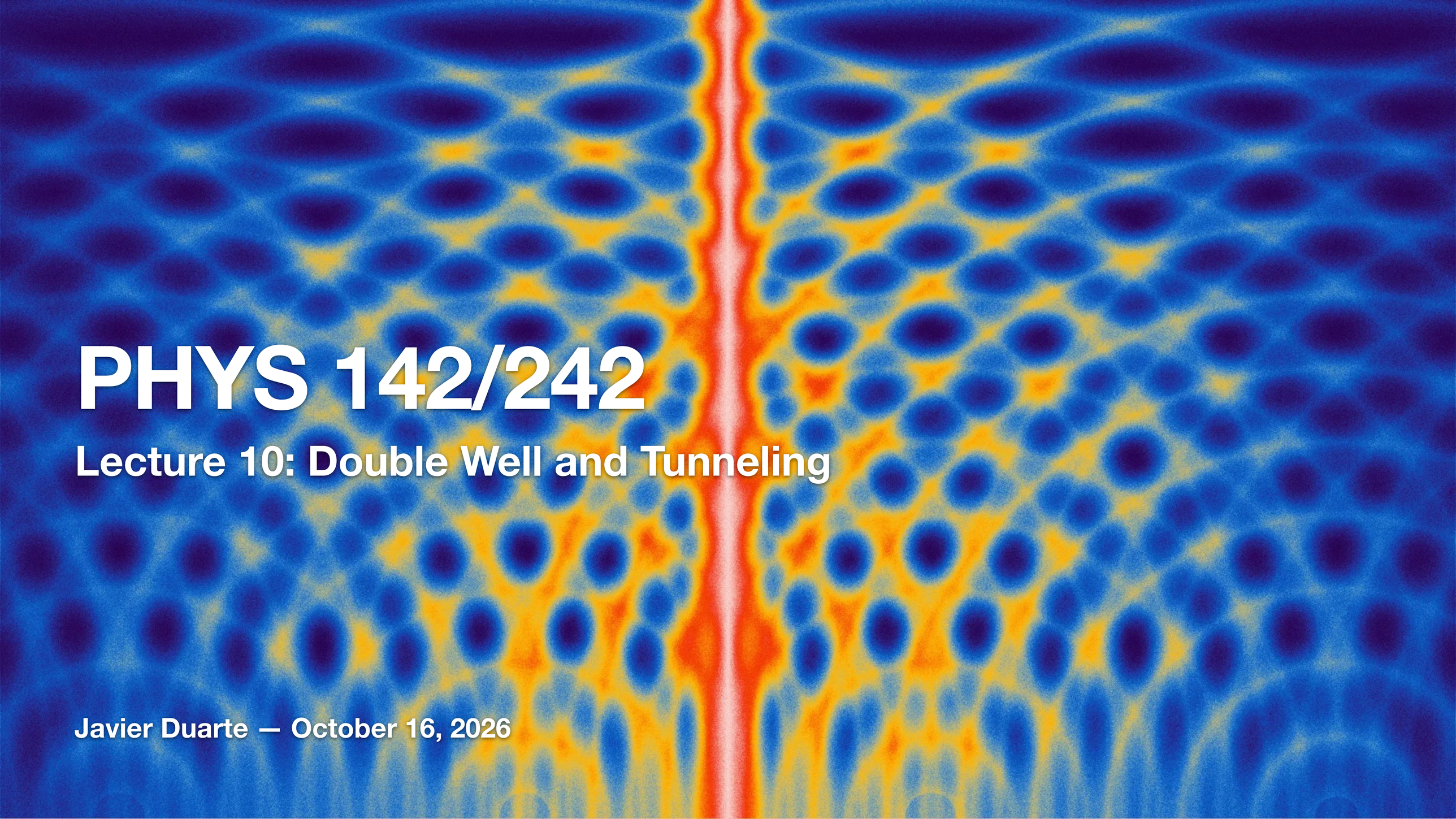


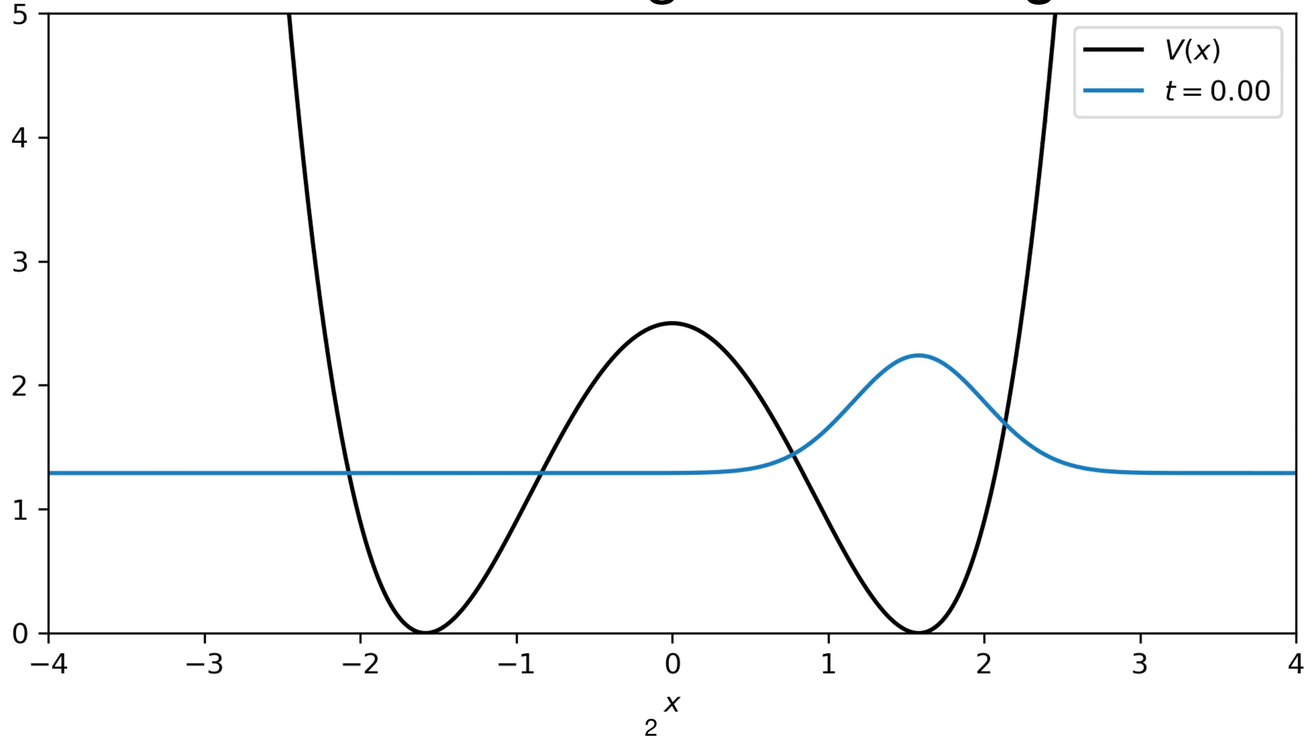
PHYS 142/242

Lecture 10: Double Well and Tunneling

Javier Duarte — October 16, 2026

Quantum tunneling

- *Quantum tunneling* is the phenomenon in which a particle passes through a potential energy barrier that, according to classical mechanics, should not be passable because it doesn't have sufficient energy to surmount the barrier
- We will analyze this phenomena in Assignment 2 using the *double well potential*
- In particular, we are interested in finding the *tunneling time*



Double well potential

Consider the double well potential

$$V(x) = \alpha x^4 - 2x^2 + \frac{1}{\alpha}$$

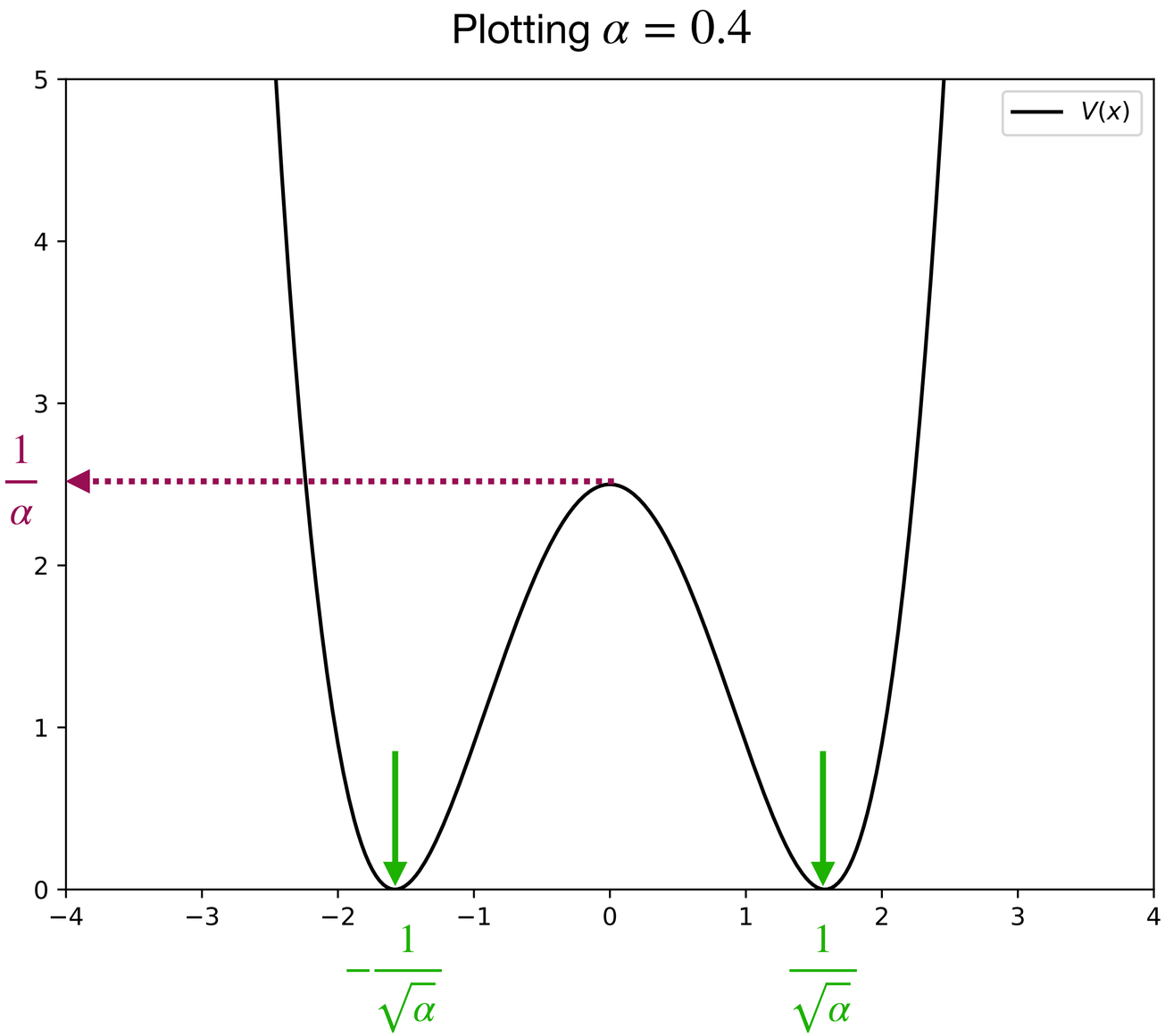
Two minima separated by a potential barrier

Barrier height: $V(0) = \frac{1}{\alpha}$

Minima:

$$V'(x_{\min}) = 4\alpha x_{\min}^3 - 4x_{\min} = 0$$

$$\Rightarrow x_{\min} = \pm \frac{1}{\sqrt{\alpha}}$$



Double well potential

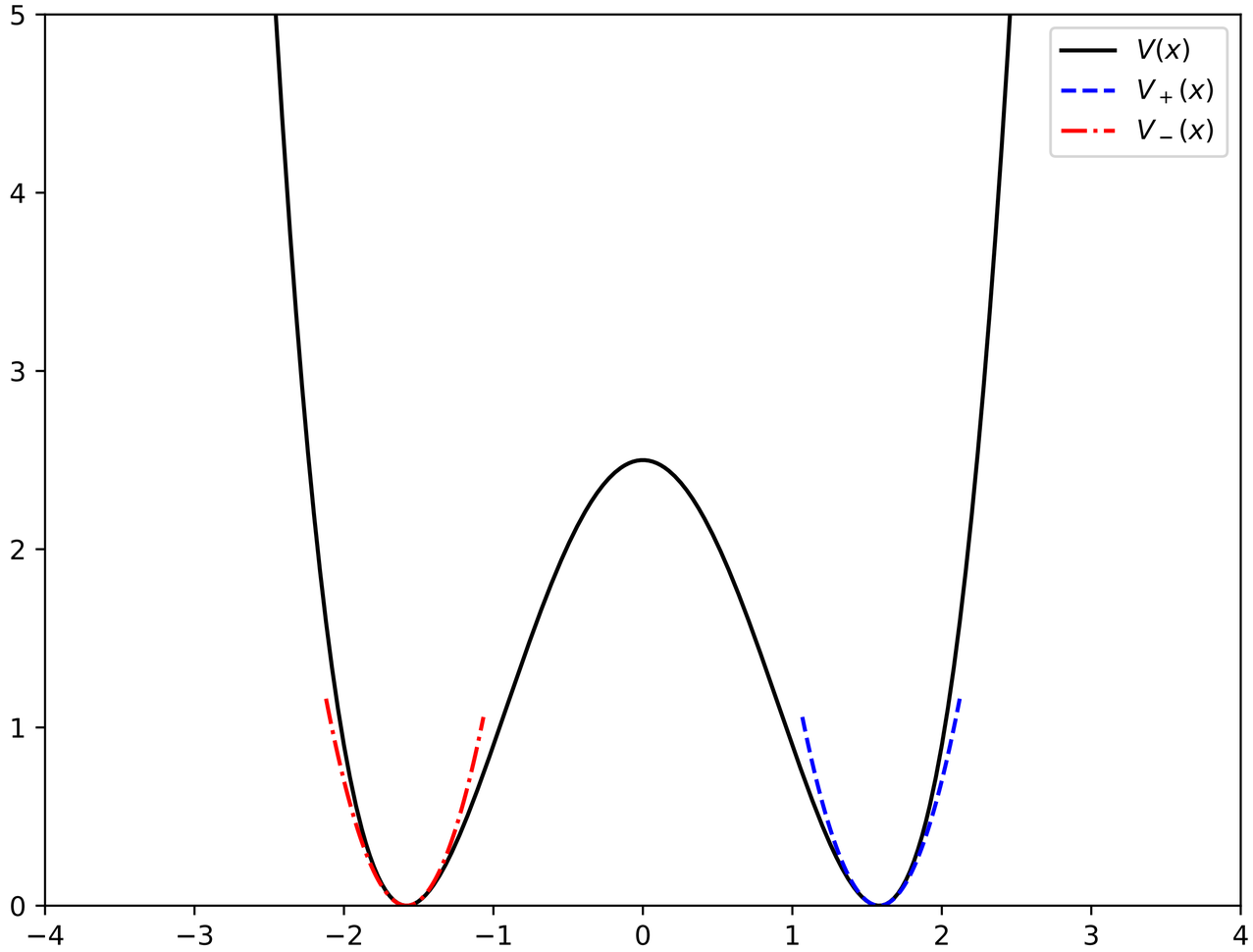
Near the minima, we approximate the potential using quadratic harmonic oscillator potentials

$$V''(x_{\min}) = 12\alpha x_{\min}^2 - 4$$
$$= 12\alpha\left(\frac{1}{\alpha}\right) - 4$$
$$= 8$$

$$V_+(x) = 4(x - x_{\min})^2$$

$$V_-(x) = 4(x + x_{\min})^2$$

Equivalent to $m = 1,\ \omega = 2\sqrt{2}$



Double well potential

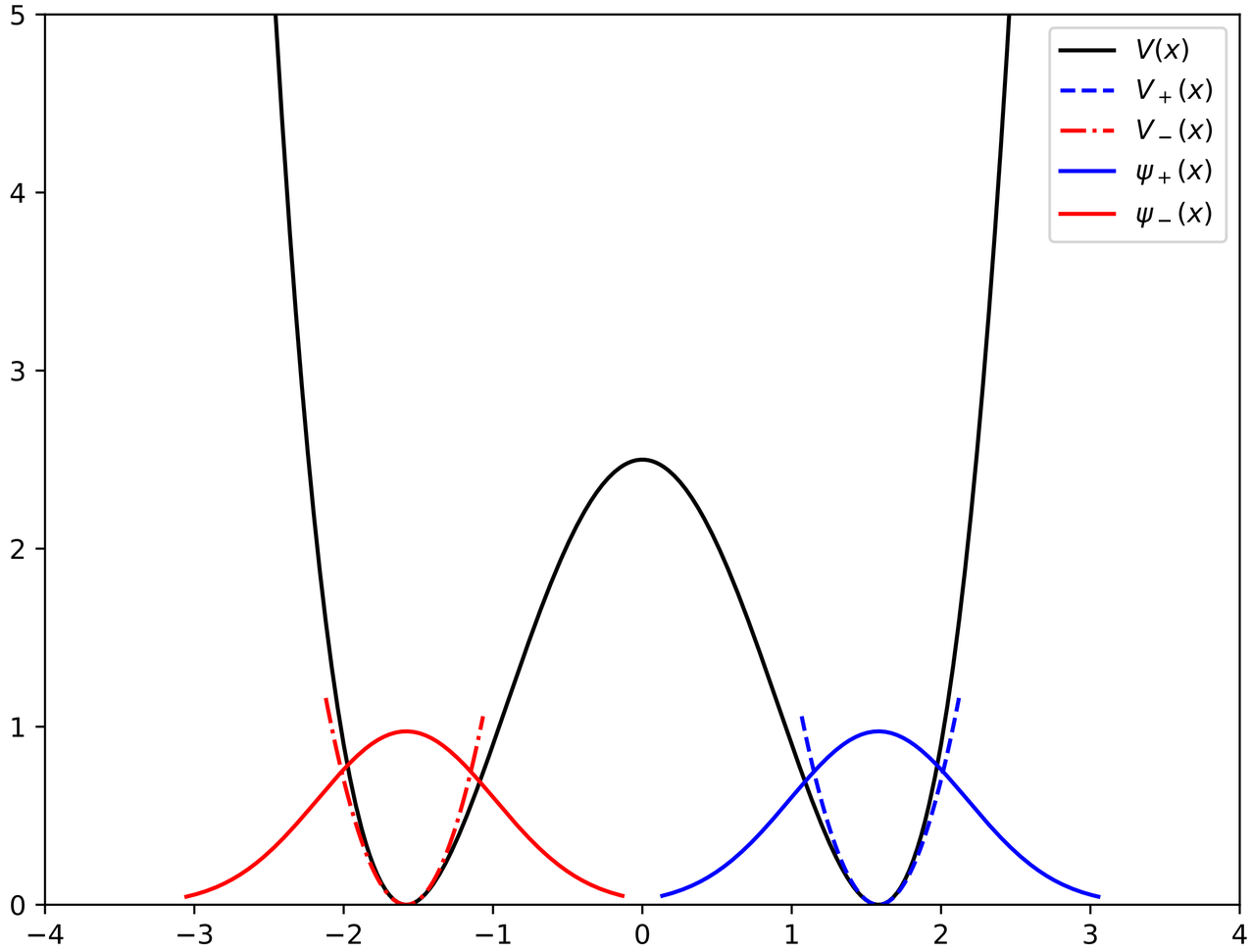
Typically, we solve the eigenvalue equation

$$-\frac{1}{2}\frac{\partial^2\psi_n(x)}{\partial x^2} + V(x)\psi_n(x) = E_n\psi_n(x)$$

But we already know the solutions for the HO

$$\psi_+(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x - x_{\min})^2\right)$$

$$\psi_-(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(x + x_{\min})^2\right)$$



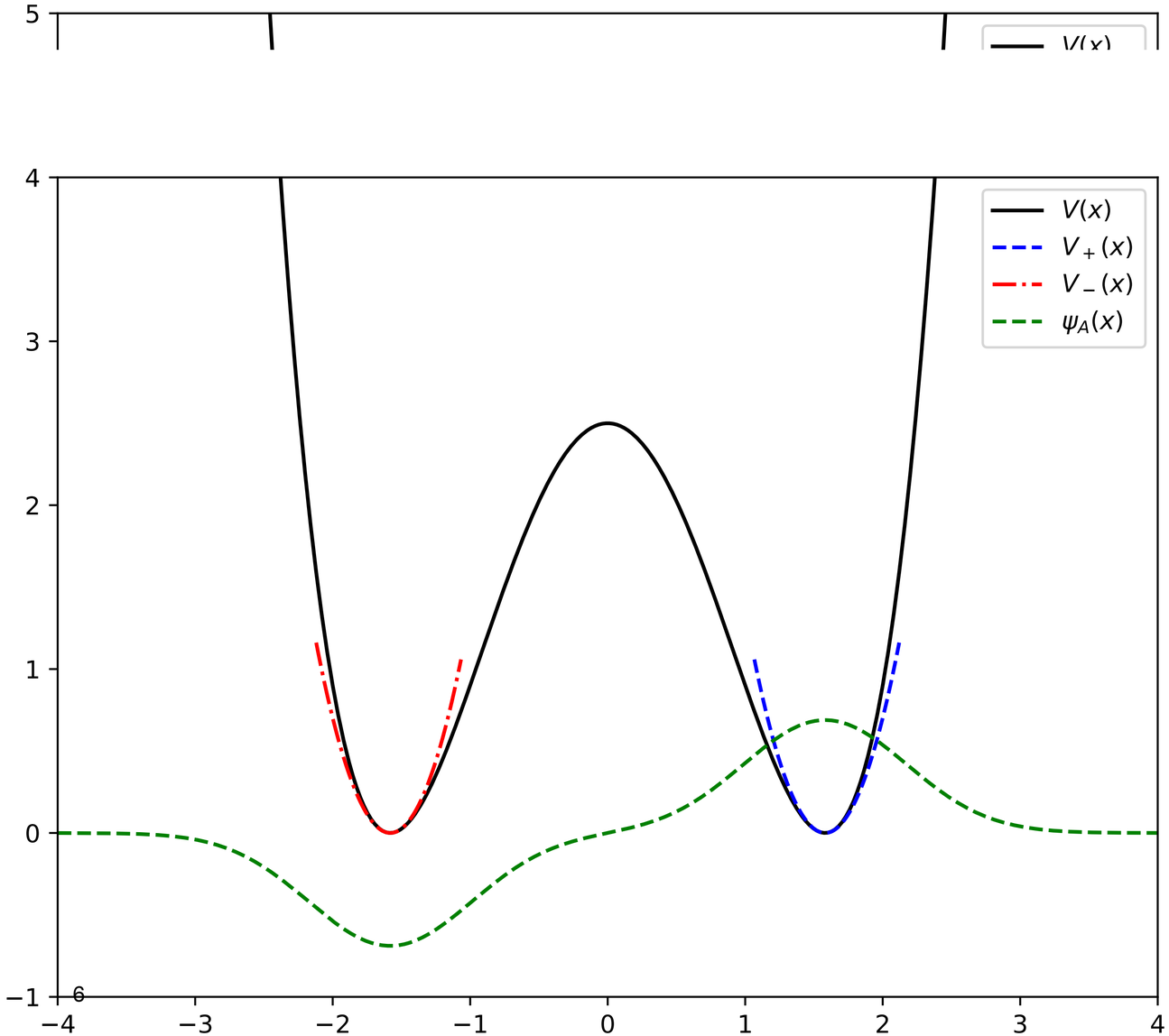
Double well potential

The ground state $\psi_0(x)$ and first excited state $\psi_1(x)$ of the full potential should be *symmetric* $\psi_0(x) = \psi_0(-x)$ and *antisymmetric* $\psi_1(x) = -\psi_1(-x)$, respectively

We can approximate them with

$$\psi_0(x) \approx \psi_S(x) = \frac{1}{\sqrt{2}}\left(\psi_+(x) + \psi_-(x)\right)$$

$$\psi_1(x) \approx \psi_A(x) = \frac{1}{\sqrt{2}}\left(\psi_+(x) - \psi_-(x)\right)$$



In-class problems

(1) Why are the solutions symmetric or anti-symmetric?

Parity! If $V(-x) = V(x)$, the Hamiltonian commutes with the parity operator P. So if $\psi(x)$ is an eigenstate, $\psi(-x)$ is an eigenstate with the same energy.

In 1D, bound states are **nondegenerate** (two solutions at the same E that both vanish at infinity must be proportional). So $\psi(-x) = c\,\psi(x)$, and since $P^2 = 1$, $c = \pm 1$. Every bound eigenstate is therefore either symmetric or antisymmetric.

(2) Why is the antisymmetric solution higher energy?

The ground state has no nodes, so it must be the even one. The odd state is forced to have a node at the barrier center. That means more curvature and more kinetic energy!

Double well potential

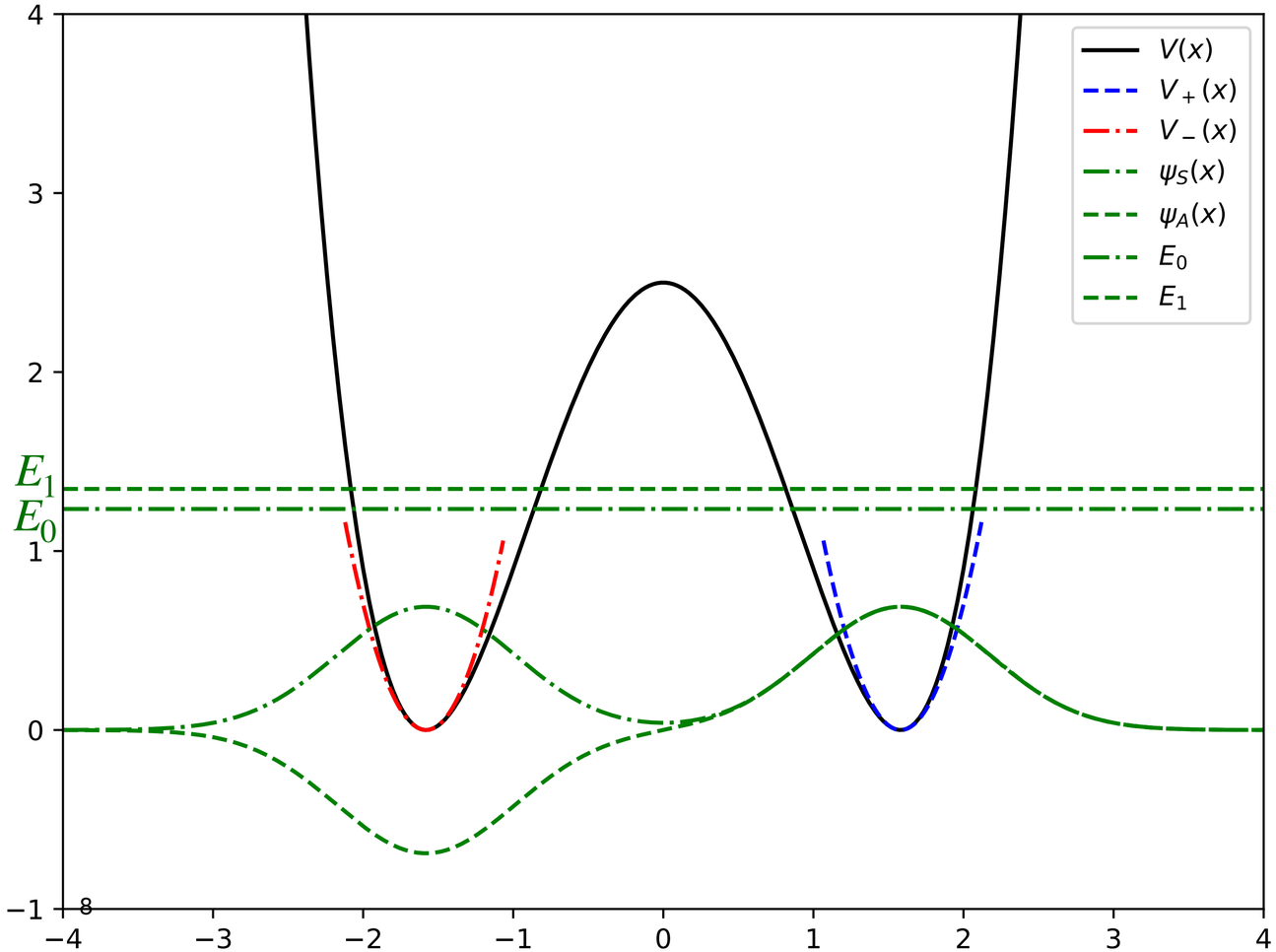
The two lowest energy eigenvalues E_0 and E_1 are nearly degenerate

$$\Delta E \equiv E_1 - E_0 \ll \frac{1}{2}(E_0 + E_1)$$

If we know the energy eigenvalues, we can time evolve

$$\psi_{\mathrm{S}}(x,t) = e^{-iE_0 t}\psi_{\mathrm{S}}(x,0)$$

$$\psi_{\mathrm{A}}(x,t) = e^{-iE_1 t}\psi_{\mathrm{A}}(x,0)$$



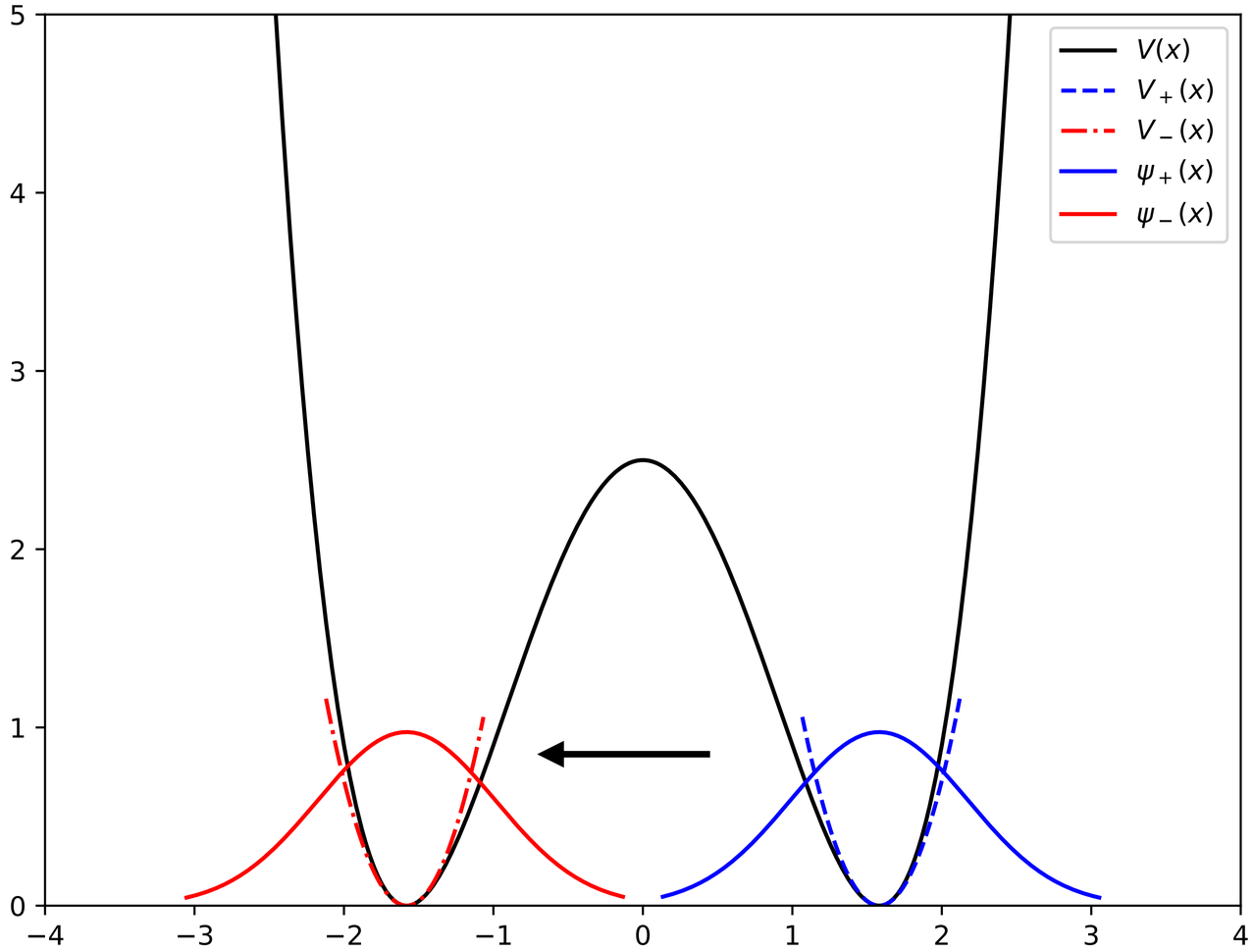
Tunneling time

Let's say we start in the right well

$$\psi(x,0) = \psi_+(x)$$

Define the tunneling time t_{tunnel} as the time it takes for the particle to fully tunnel into the left well

$$\psi(x, t_{\text{tunnel}}) \propto \psi_-(x)$$



Tunneling time

Note:

$$\psi(x,0) = \frac{1}{\sqrt{2}}\left(\psi_S(x,0) + \psi_A(x,0)\right) = \psi_+(x)$$

Time evolving it we get

$$\psi(x,t) = \frac{1}{\sqrt{2}}\left(e^{-iE_0 t}\psi_S(x,0) + e^{-iE_1 t}\psi_A(x,0)\right)$$

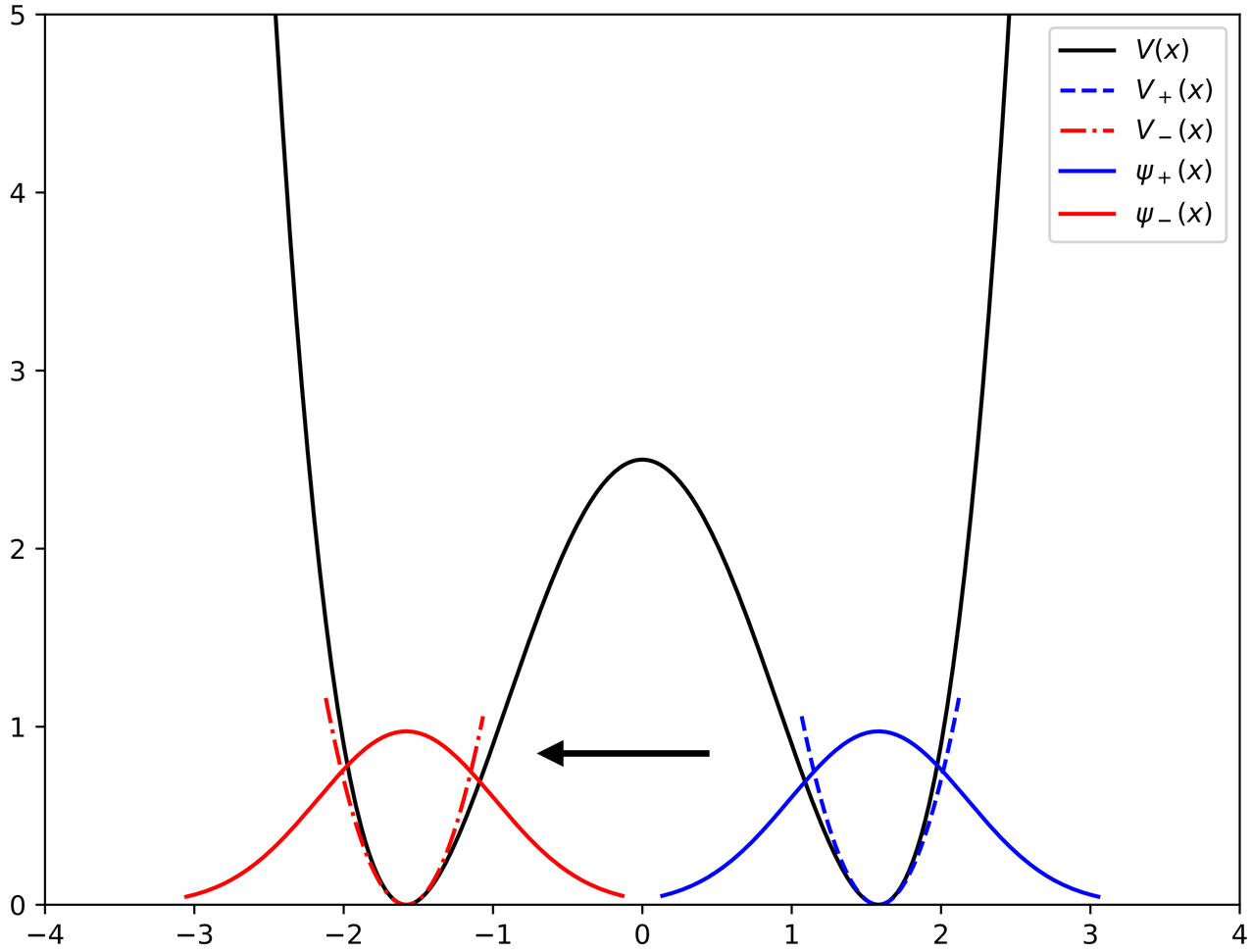
$$= \frac{1}{\sqrt{2}}e^{-iE_0 t}\left(\psi_S(x,0) + e^{-i\Delta E t}\psi_A(x,0)\right)$$

We want

$$\psi(x,t_{\text{tunnel}}) \propto \psi_-(x) = \frac{1}{\sqrt{2}}\left(\psi_S(x,0) - \psi_A(x,0)\right)$$

So the sign has to flip

$$e^{-i\Delta E t_{\text{tunnel}}} = -1 \text{ or } t_{\text{tunnel}} = \pi/\Delta E$$



Numerical results

K. Banerjee and S. P. Bhatnagar, "Two-well oscillator", Phys. Rev. D 18, 4767 (also uploaded to Canvas) numerically calculate the ground state and first excited state energy eigenvalues

Note: different convention $4\lambda = \alpha$ so $\lambda = 0.1$ corresponds to $\alpha = 0.4$

So for $\alpha = 0.4$, we have $E_0 = 1.2345$, $E_1 = 1.3469$, and $\Delta E = 0.1124$

$$t_{\text{tunnel}} = \pi/\Delta E = 27.94$$

TABLE I. Eigenvalues of the two-well oscillator in the small-λ regime. $\epsilon_n(\lambda)$ are the computed exact eigenvalues of the energy-shifted operator $H(1, \lambda) + (1/4\lambda)$, which is positive definite.

λ	ϵ_0 ϵ_1	ϵ_2 ϵ_3
0.01	1.404 048 605 297 7[a] 1.404 048 605 297 7	4.170 193 605 999 3 4.170 193 605 999 3
0.02	1.393 527 585 044 2 1.393 527 587 151 0	4.092 028 112 820 5 4.092 028 608 428 7
0.03	1.382 601 444 053 8 1.382 605 783 831 4	4.006 049 199 465 7 4.006 655 466 749 5
0.04	1.371 122 236 557 5 1.371 308 461 612 9	3.901 359 951 813 1 3.918 263 337 997 1
0.05	1.358 422 103 747 8 1.360 133 597 773 3	3.746 917 080 727 9 3.848 838 300 057 4
0.07	1.323 374 074 208 5 1.343 365 616 287 4	3.342 216 720 258 7 3.833 129 937 607 9
0.10	1.234 507 162 786 0 1.346 940 868 922 5	3.009 488 545 436 2 4.043 546 039 767 6
0.15	1.062 499 247 956 5 1.421 086 890 539 3	3.033 667 276 570 6 4.589 838 495 543 4
0.17	1.007 165 158 778 7 1.464 225 132 421 2	3.118 337 642 119 7 4.816 923 221 196 9
0.20	0.941 750 342 076 9 1.535 530 204 085 8	3.270 377 801 715 3 5.148 274 740 096 0

[a] Since near the minima the potentail function $\sim 2x^2 + O(\lambda^{1/2}x^3)$, $\epsilon_0 \to \sqrt{2}$ (ground-state energy in a potential $2x^2$) as $\lambda \to 0$. We find $\epsilon_0(\lambda = 0.001) = 1.413\,211\,965\,792$.

Quantum tunneling

- $t_{\text{tunnel}} = \pi/\Delta E = 27.94$
- Assignment 2 is to show this with Feynman path integral approach

