PHYS 142/242

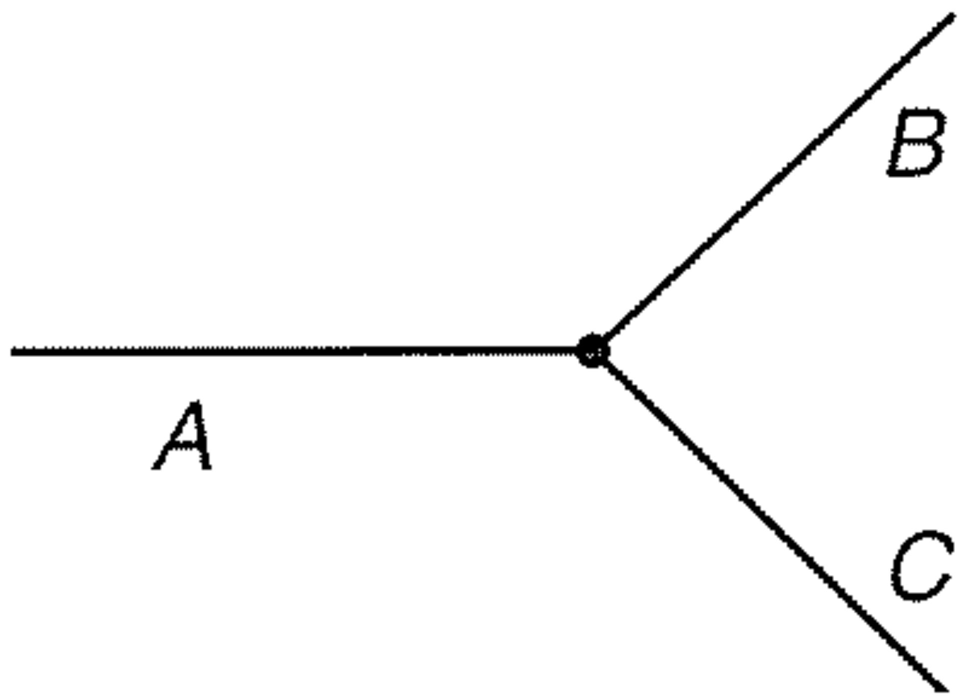
Lecture 20: Particle Physics & VEGAS (Part 3)

Javier Duarte — February 28, 2025

Course Evaluations!

- Sudent Evaluations of Teaching (SET) for Winter 2025 will be available to students from:
    - Monday, March 3 to Saturday, March 15 at 8:00 AM.
- **Students must complete their evaluations BEFORE 8AM on Saturday, March 15 at 8:00 AM. No exceptions.**
- Students will access their SETs from the Evaluations site. Students will receive an email directing them where to access their evaluations. They will also receive an email confirmation for each SET they complete.
- If we reach above 80% submitted evaluations, extra credit on final projects!

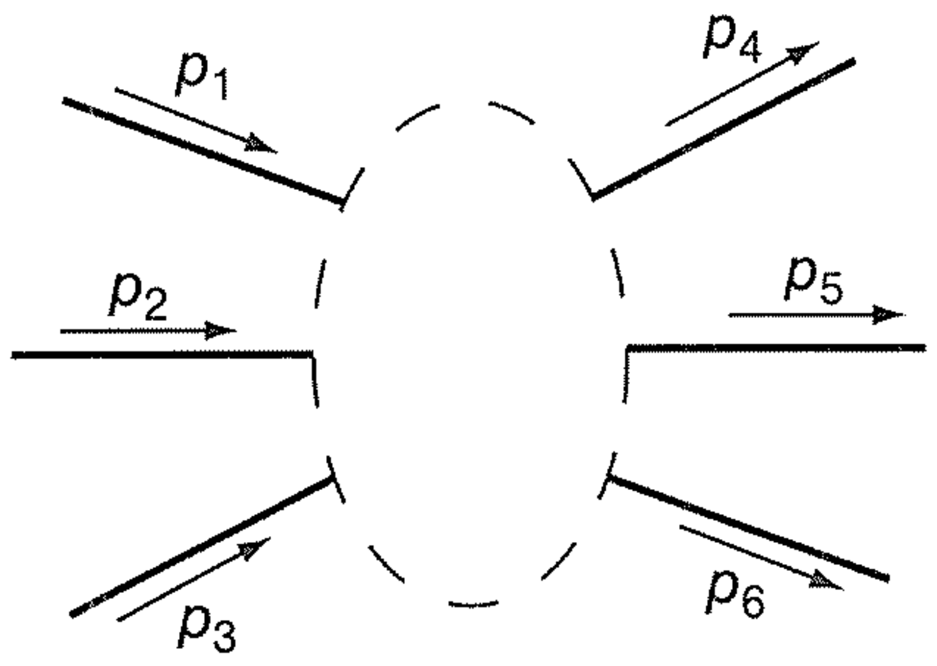
Feynman rules for ABC toy theory



1. Label incoming/outgoing 4-momentum $p_1, p_2, \ldots, p_n$ and internal 4-momenta $q_1, q_2, \ldots$

2. Vertex factors: For each vertex, write $-ig$

3. Propagators: For each internal line, write $\dfrac{i}{q_j^2 - m_j^2}$



4. Conservation of energy & momentum: For each vertex, write $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$

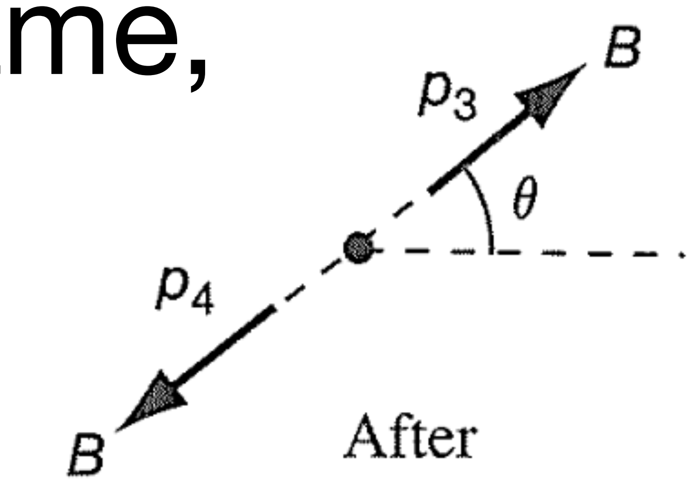
5. Integrate over internal momentum: For each internal line, write $\dfrac{1}{(2\pi)^4} d^4 q_j$

6. Cancel the delta function $(2\pi)^4 \delta^4(p_1 + p_2 + \ldots + p_n)$ and multiply by i to get $\mathscr{M}$

Scattering: $A + A \to B + B$

Assuming massless particles and CM frame,

Before



After

$$\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2$$

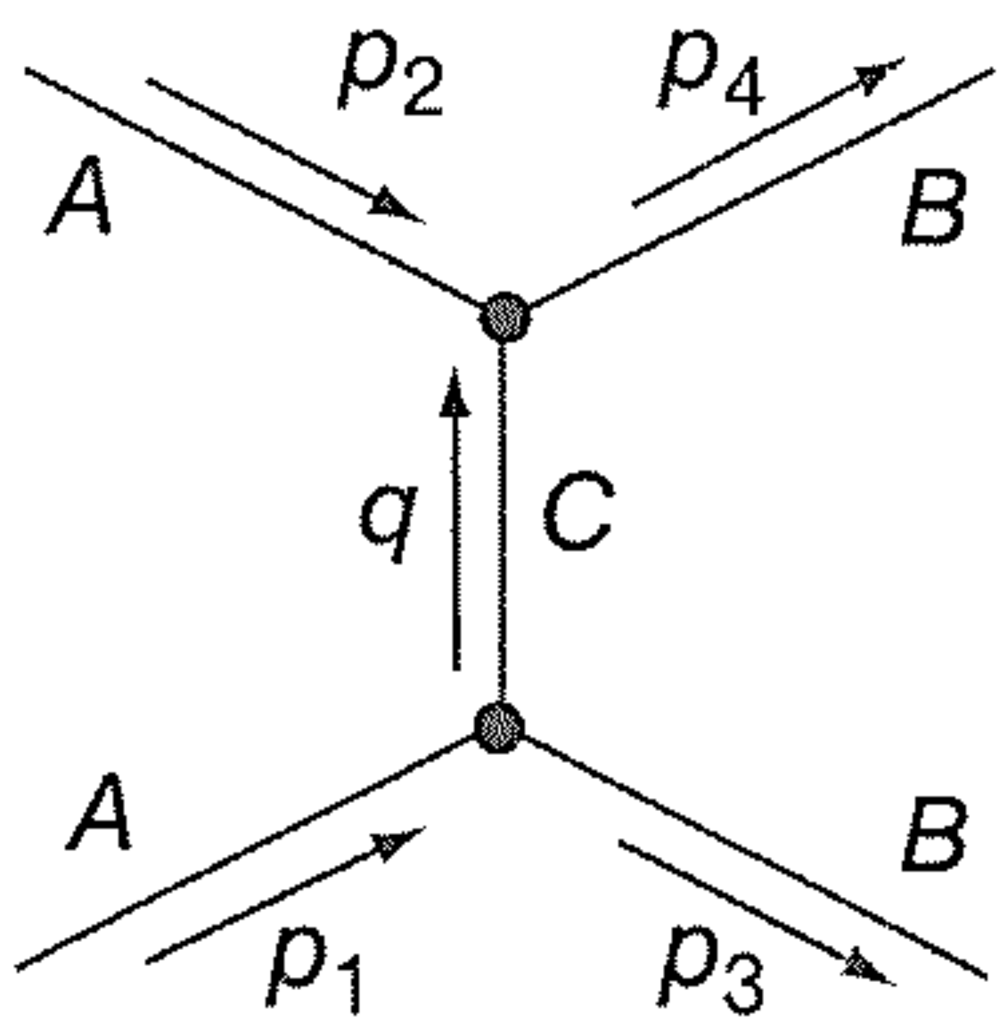
$$\mathscr{M}_1 = \int i(-ig)^2 \frac{i}{q^2} \times (2\pi)^4 \delta^4(p_1 - q - p_3)$$

$$\times (2\pi)^4 \delta^4(p_2 + q - p_4) \frac{1}{(2\pi)^4} d^4 q$$

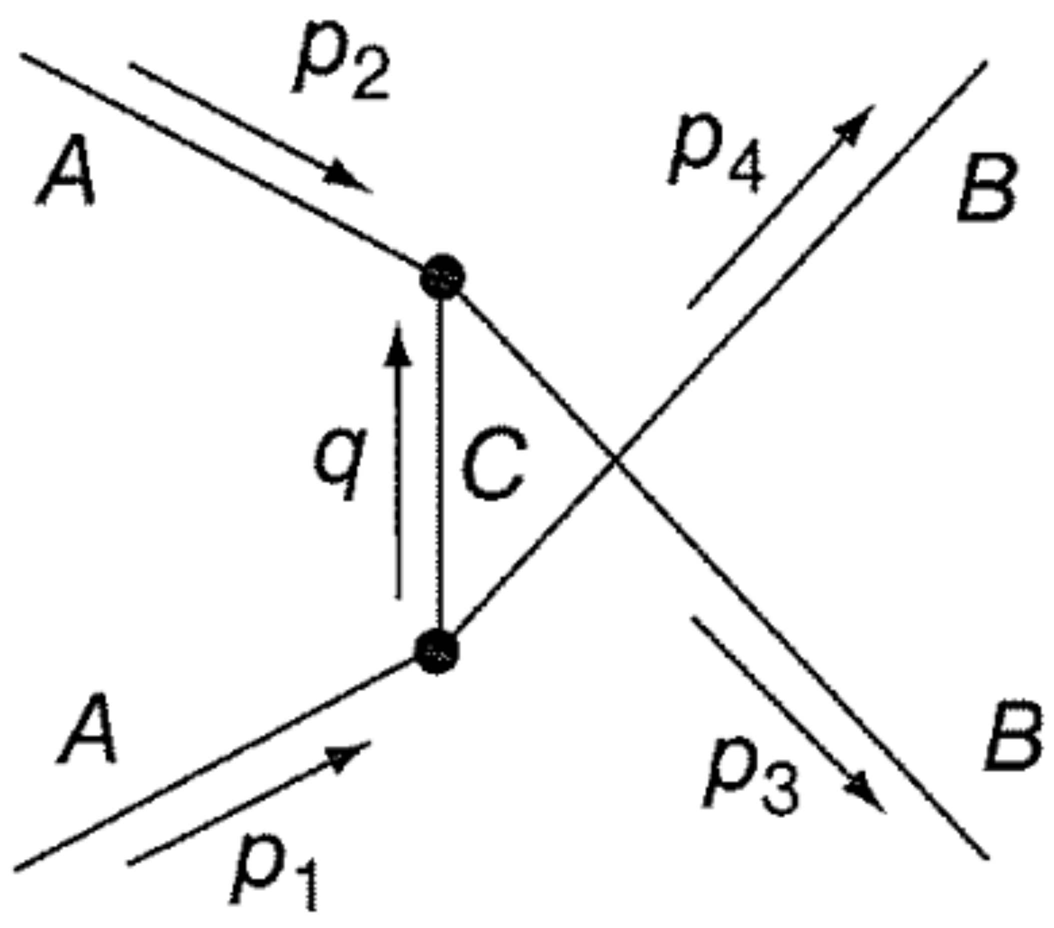
$$= \frac{g^2}{(p_1 - p_3)^2} = \frac{g^2}{-2p_1 \cdot p_3} = \frac{g^2}{-2\boldsymbol{p}^2(1 - \cos\theta)}$$

$$\mathscr{M}_2 = \frac{g^2}{(p_1 - p_4)^2} = \frac{g^2}{-2p_1 \cdot p_4} = \frac{g^2}{-2\boldsymbol{p}^2(1 + \cos\theta)}$$

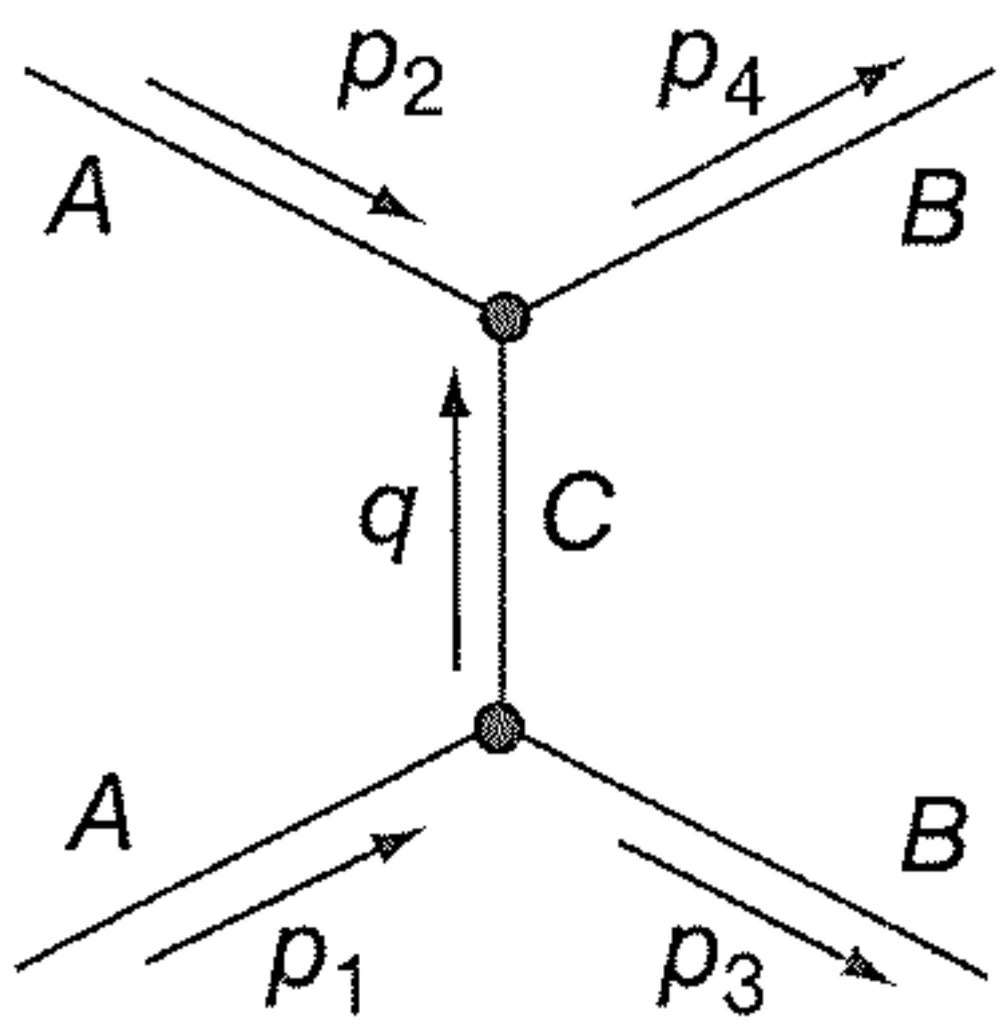
$\mathscr{M}_1$



$\mathscr{M}_2$



Scattering: $A + A \to B + B$



Matrix element is

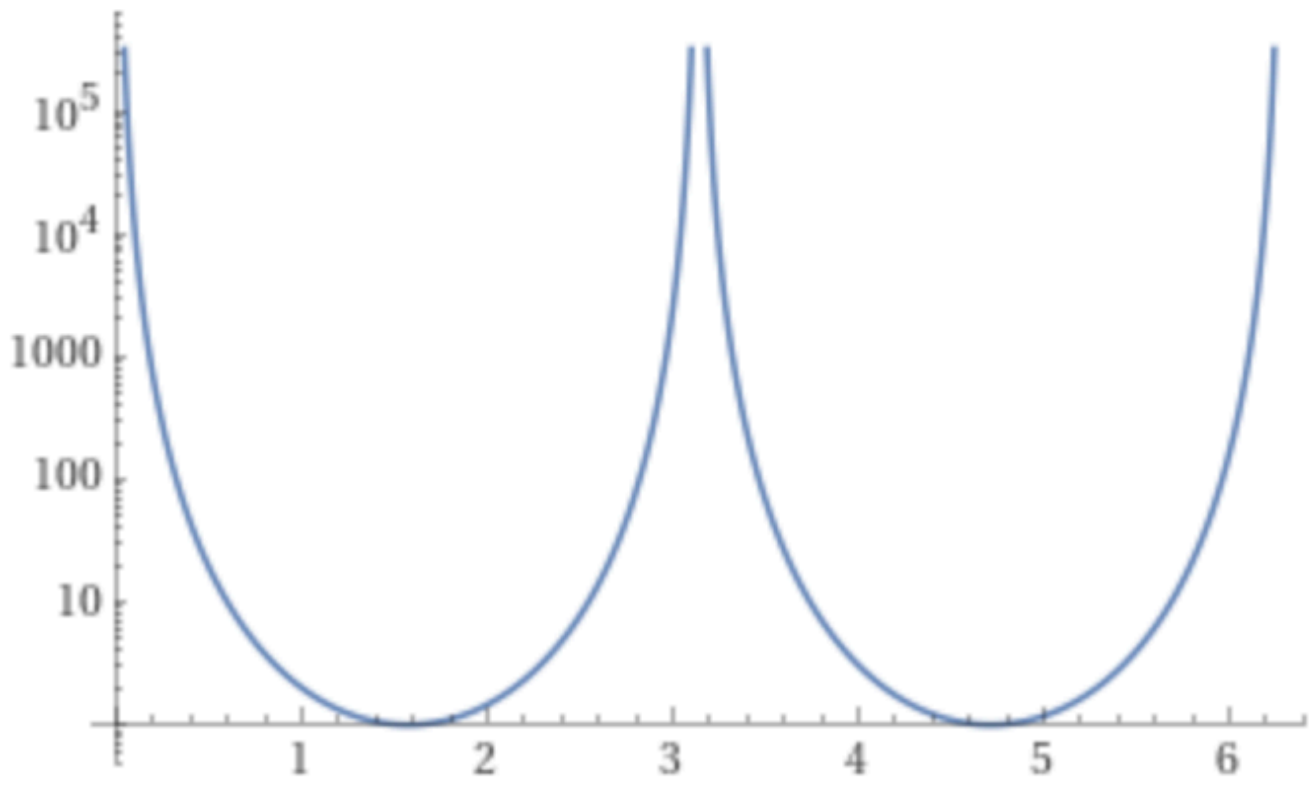
$$\mathscr{M} = -\frac{g^2}{\boldsymbol{p}^2(1 - \cos^2\theta)} = -\frac{g^2}{\boldsymbol{p}^2 \sin^2\theta}$$

Differential cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathscr{M}|^2}{(E_1 + E_2)^2} \frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|} = \frac{1}{2}\left(\frac{g^2}{16\pi E \boldsymbol{p}^2 \sin^2\theta}\right)^2$$

To get the total cross section, we would integrate this

$$\sigma = \int d\Omega \frac{d\sigma}{d\Omega} = \int d\theta d\phi \sin\theta \frac{d\sigma}{d\Omega}$$



Monte Carlo integration

Basis of all Monte Carlo integration is the simple observation: the value of an integral can be recast as the average of the integrand:

$$I = \int_{x_1}^{x_2} f(x)dx = (x_2 - x_1)\langle f(x) \rangle$$

So, if we take N values of x distributed uniformly in (x_1, x_2), then the average of $f(x)$ will be a good estimator

$$I \approx I_N = (x_2 - x_1)\frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

Monte Carlo uncertainty

Introducing the weight $W_i = (x_2 - x_1)f(x_i)$,

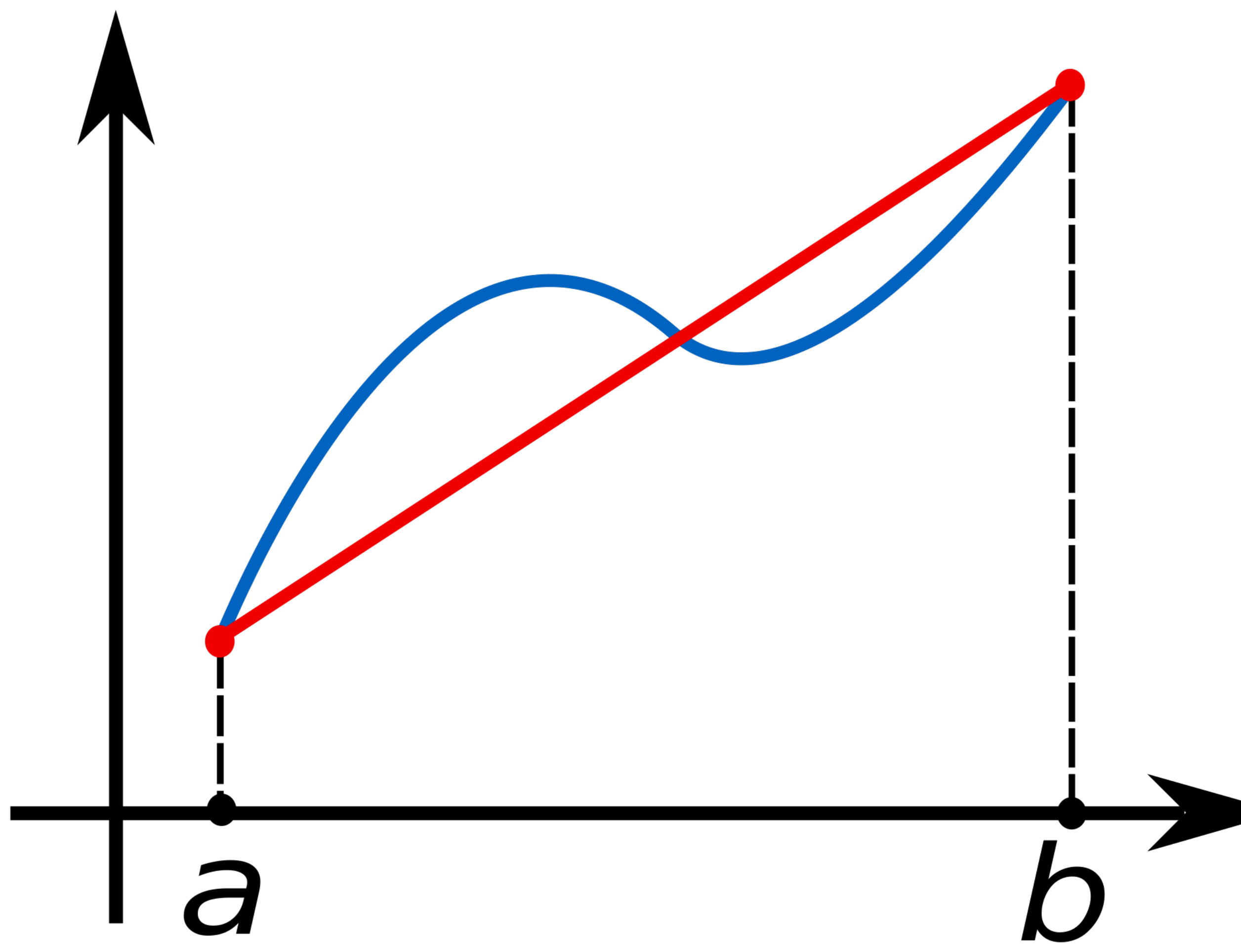
$$I_N \equiv \frac{1}{N}\sum_{i=1}^{N} W_i$$

and

$$V_N \equiv \sigma^2 = \frac{1}{N}\sum_{i=1}^{N} W_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N} W_i\right)^2$$

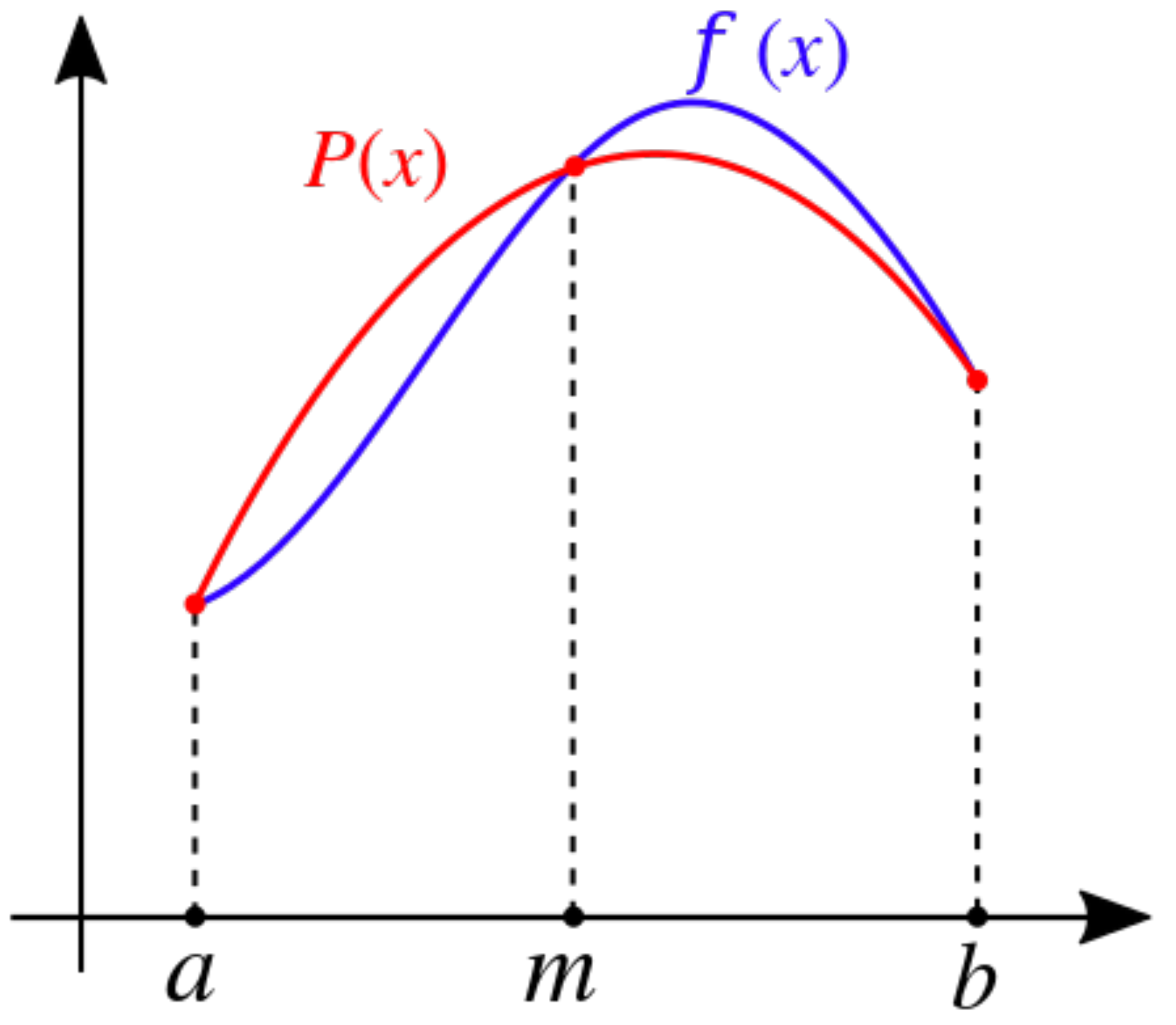
Then $\sigma_{\mathrm{MC}} = \sqrt{V_N/N} = \sigma/\sqrt{N}$ and $I \approx I_N \pm \sqrt{\frac{V_N}{N}}$

Convergence in d dimensions



In d dimensions,

- MC integration still converges $\propto 1/\sqrt{N}$
- Trapezoidal rule converges $\propto 1/N^{2/d}$
- Simpson's rule converges $\propto 1/N^{4/d}$



Particle physics: many dimensions

$$\sigma = \frac{1}{2E_{\mathrm{CM}}} \int f(x_1) f(x_2) \, |\mathscr{M}|^2 \, dx_1 dx_2 d^3p_1 d^3p_2 \cdots d^3p_n \delta^4(P - p_1 - p_2 - \cdots - p_n)$$

MC advantages and disadvantages

Disadvantages of MC

- Slow convergence in few dimensions

Advantages of MC

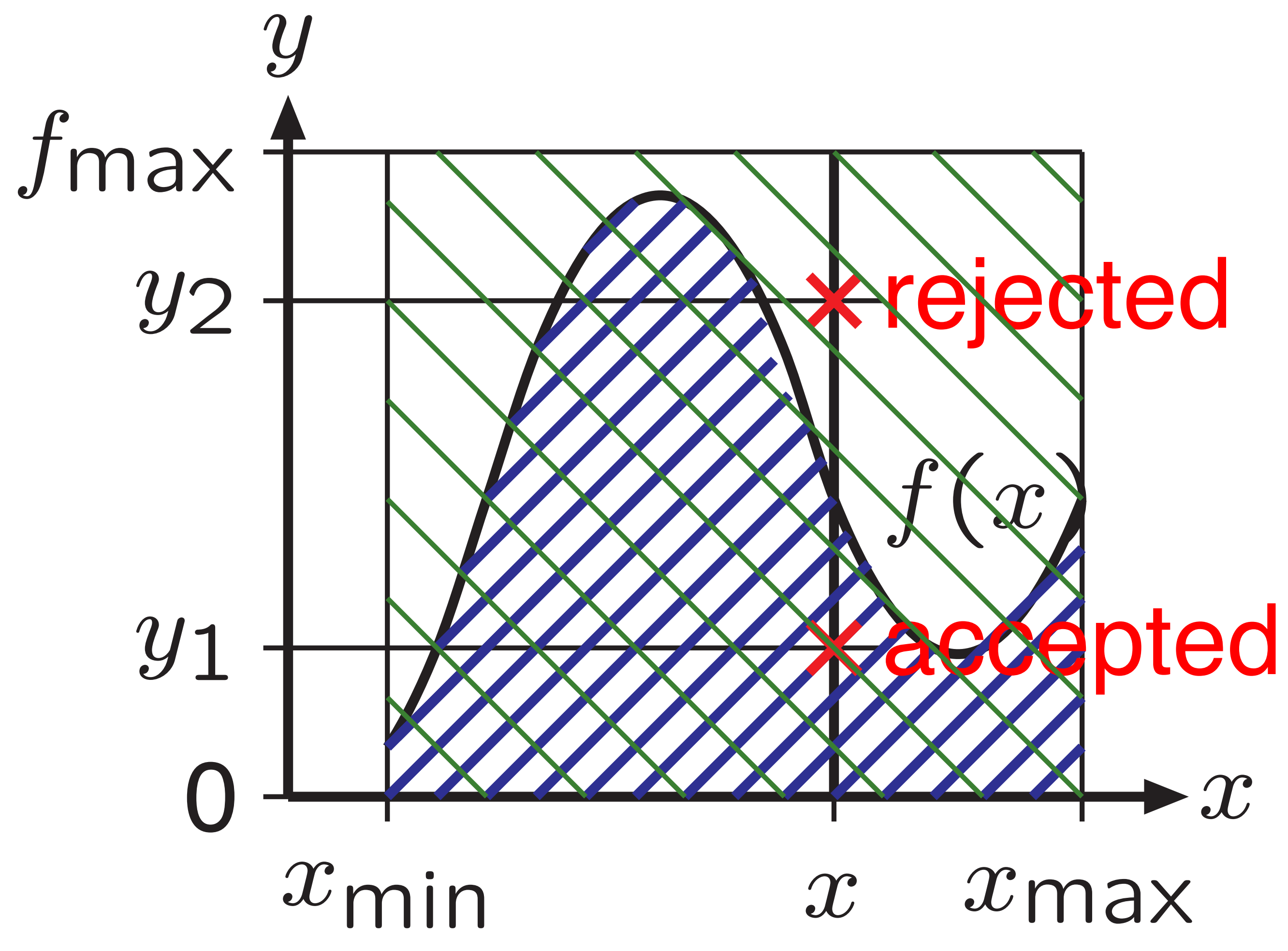
- Fast convergence in many dimensions
- Arbitrarily complex integration regions (finite discontinuities not a problem)
- Few points needed to get first estimate (“feasibility limit”)
- Every additional point improves accuracy (“growth rate”)
- Easy error estimate

MC for event generators

Up to here, only considered MC as a numerical integration method

If function being integrated is a probability density (positive definite), can convert it to a simulation of physical process = an event generator

Simple example: $\sigma = \int_0^1 \frac{d\sigma}{dx} dx$



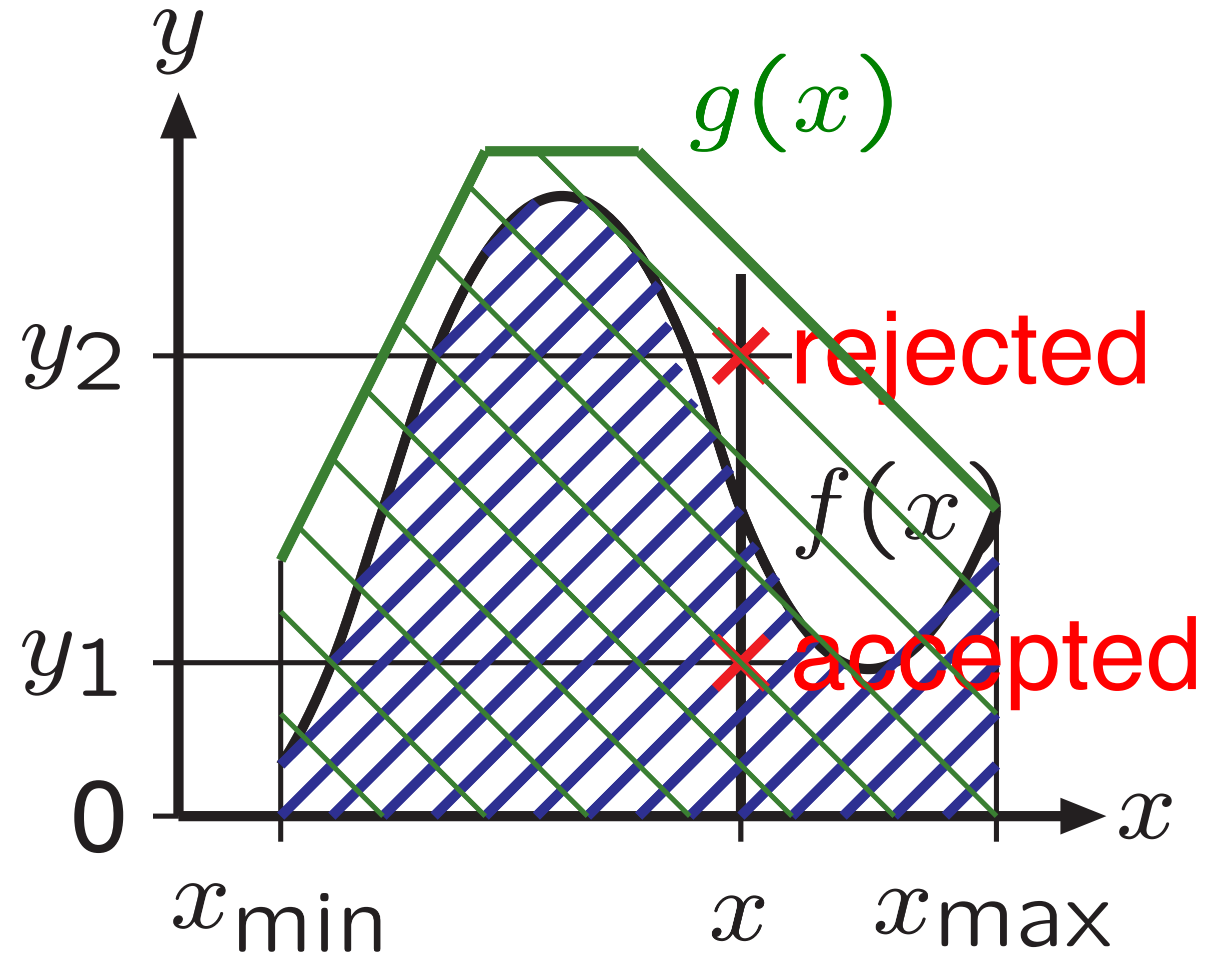
Weighted events: generate events x with weights $d\sigma/dx$

Unweighted events: generate events x by keeping them with probability $(d\sigma/dx)/(d\sigma/dx)_{\max}$ and giving them all weight $\langle d\sigma/dx \rangle$ calculated over all generated events (not just accepted ones)

Importance sampling

To improve slow convergence, can use *importance sampling*: sample more often when probability is high and less often when probability is low

Consider integral in one dimension $I = \int_0^1 f(x)dx$; We want to get the best estimate given M integrand evaluations (total samples)



If we have PDF $p(x)$ where $p(x) > 0$ and $\int_0^1 p(x)dx = 1$

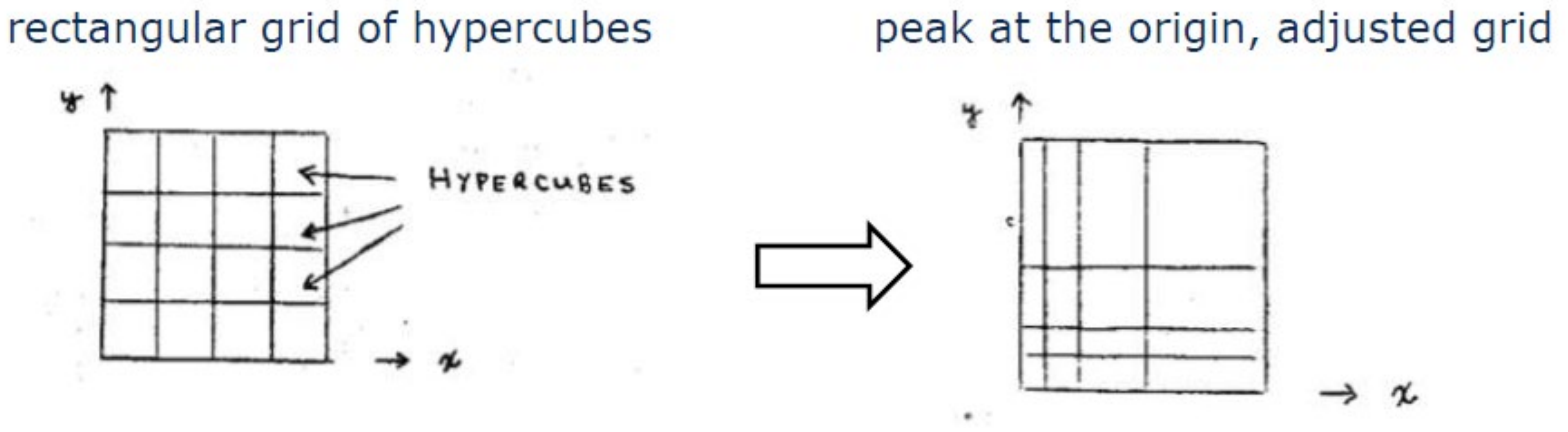
Optimal variance when

$$p(x) = \frac{|f(x)|}{\int_0^1 |f(x)|dx}$$

VEGAS

Exploratory phase:

- subdivide integration space into rectangular grid
- perform integration in each subspace
- adjust grid according to dominant contributions
- integrate again, approximate optimal

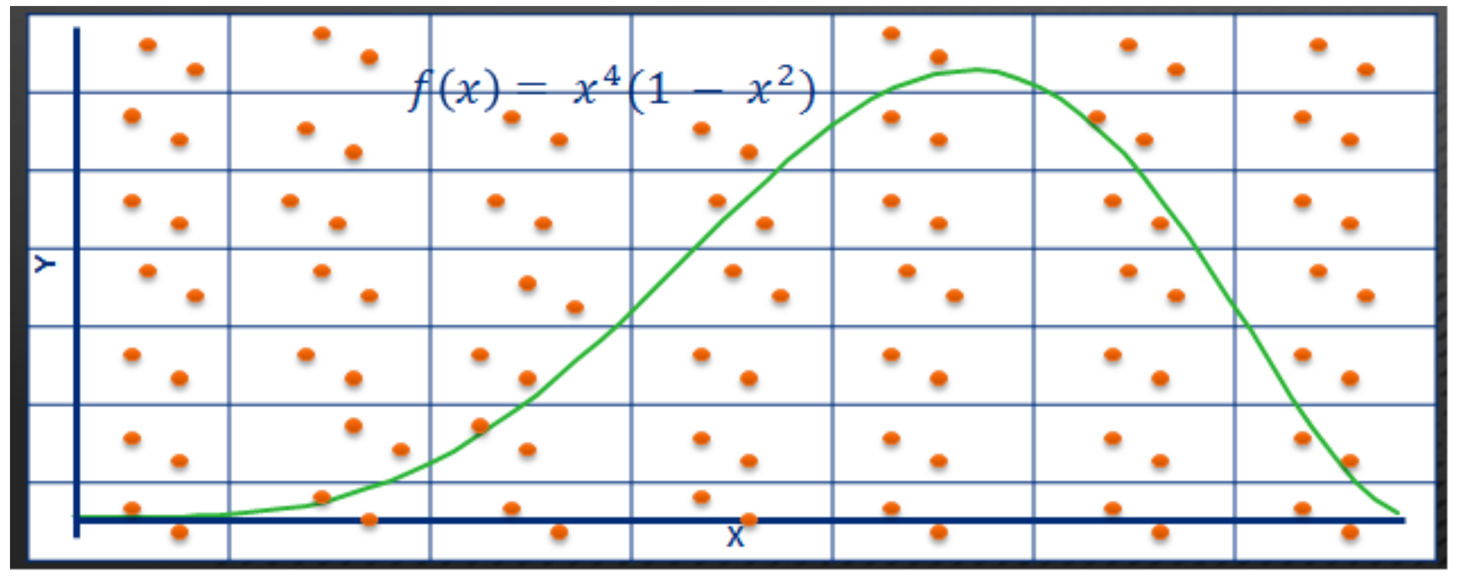


Evaluation phase:

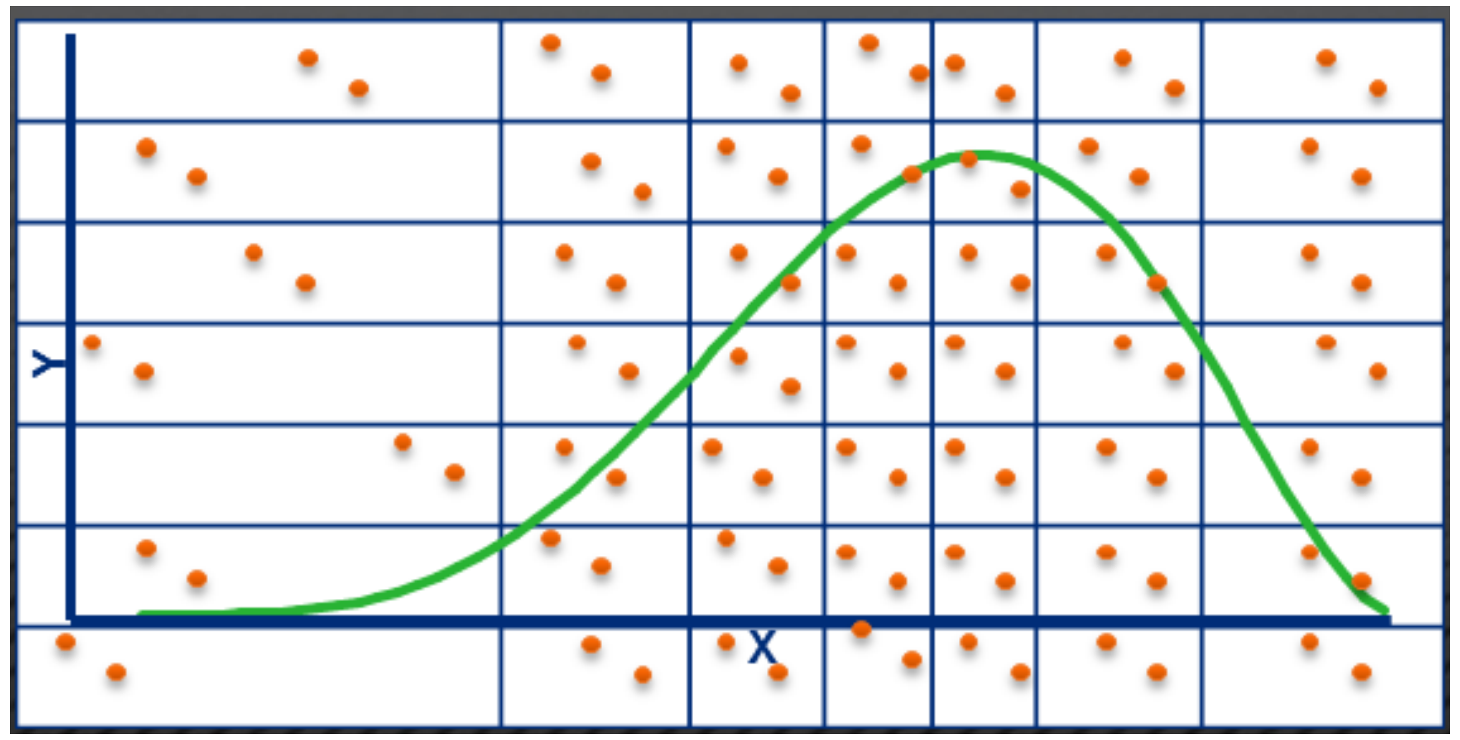
- integrate with high precision and optimized frozen grid or efficiently generate events using optimized frozen grid

VEGAS algorithm (1)

VEGAS algorithm begins by approximating $p(x)$ as a step function with N steps, starting with uniform probability for all steps: $0 = x_0 < \cdots < x_N = 1$, $\Delta x_i = x_i - x_{i-1}$



$$p(x) = \frac{1}{N\Delta x_i}, \quad x_i - \Delta x_i \le x < x_i, \quad i = 1, \ldots, N$$



where

$$\sum_{i=1}^{N} \Delta x_i = 1$$

Each interval gets approximately the same number of samples $n_i \approx M/N$.

For each iteration, we adjust the intervals $\{x_i, \Delta x_i\}$

VEGAS algorithm (2)

Consider the contribution to the integral of the function in a given interval

$$\bar{f}_i = \sum_{x \in [x_i - \Delta x_i, x_i]} |f(x)|$$

Define $m_i = \text{int}\left(K \frac{\bar{f}_i \Delta x_i}{\sum_j \bar{f}_j \Delta x_j}\right)$ where typically $K \approx 1000$

Divide each interval Δx_i into $m_i + 1$ subintervals

Since the total number of subintervals is now $\gg N$, the subintervals are combined into N groups, which define the new set of intervals

Optimal to combine them so that $\bar{f}_i$ are approximately equal across all N

VEGAS algorithm (3)

To avoid rapid, destabilizing changes in the grid, it is better to damp the subdivisions using

$$m_i = \mathrm{int}\left(K \left[\frac{\frac{\bar{f}_i \Delta x_i}{\sum_j \bar{f}_j \Delta x_j} - 1}{\log\left(\frac{\bar{f}_i \Delta x_i}{\sum_j \bar{f}_j \Delta x_j} \right)} \right]^{\alpha} \right)$$

α typically set between 0.2 and 2

VEGAS error analysis

To obtain a cumulative estimate of the integral and its standard error from each iteration j

$$\bar{I} = \frac{\sum_j I_j/\sigma_j^2}{\sum_j 1/\sigma_j^2}$$

$$\sigma_{\bar{I}} = \left(\sum_j \frac{1}{\sigma_j^2} \right)^{-1/2}$$

Also good sanity check is that $\chi^2 = \sum_j \frac{(I_j - \bar{I})^2}{\sigma_j^2} \approx N_{\text{iter}} - 1$

VEGAS higher dimensions

In 2-d we write $t = \int_0^1 dx \int_0^1 dy\, f(x,y) = \int_0^1 dx\, \rho_x(x) \int_0^1 dy\, \rho_y(y) \frac{f(x,y)}{\rho_x(x)\rho_y(y)}$

Divide $x-$axis into N equal segments $0 = x_0 < x_1 < \cdots < x_N = 1$, $\Delta x_i = x_i - x_{i-1}$

Divide $y-$axis into N equal segments $0 = y_0 < y_1 < \cdots < y_N = 1$, $\Delta y_i = y_i - y_{i-1}$

$\rho_x(x) = \frac{1}{N\Delta x_i}$ for $x_i - \Delta x_i \le x < x_i$, i=1,2,...,$N$

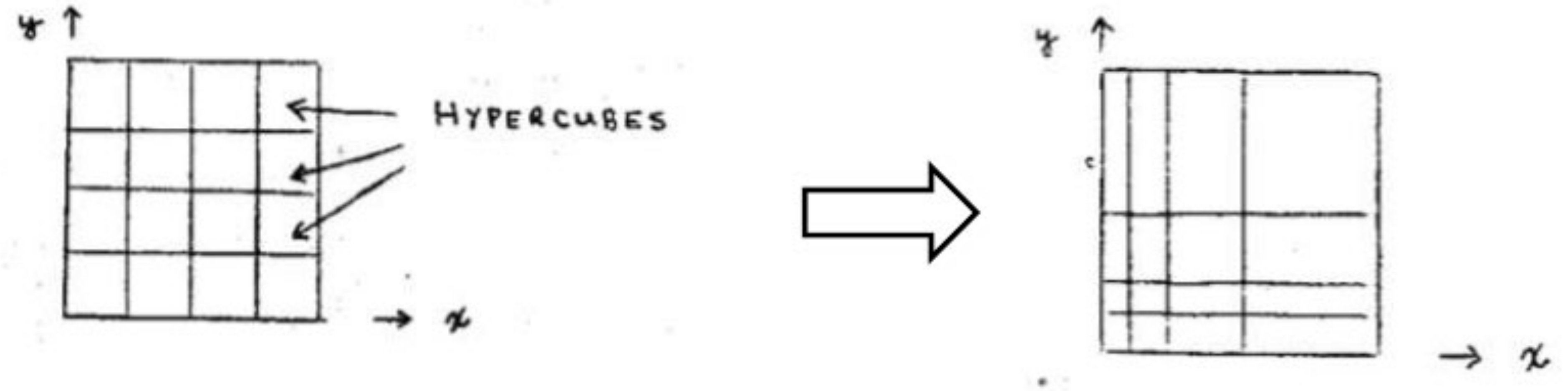
$\rho_y(y) = \frac{1}{N\Delta y_i}$ for $y_i - \Delta y_i \le y < y_i$, i=1,2,...,N

1-d algorithm applied along $x-$axis with

$$(\bar{f}_i)^2 = \sum_{x_i - \Delta x \le x < x_i} \sum_{0<y<1} \frac{f^2(x,y)}{\rho_y^2(y)} \propto \frac{1}{\Delta x_i} \int_{x_i - \Delta x_i}^{x_i} dx \int_0^1 dy \frac{f^2(x,y)}{\rho_y(y)}$$

1-d algorithm applied along y$-$axis with

$$(\bar{f}_i)^2 = \sum_{y_i - \Delta y \le y < y_i} \sum_{0<x<1} \frac{f^2(x,y)}{\rho_x^2(x)} \propto \frac{1}{\Delta y_i} \int_{y_i - \Delta y_i}^{y_i} dy \int_0^1 dx \frac{f^2(x,y)}{\rho_x(x)}$$



Easy to generalize, but assumption is $f(x, y)$ can be reasonably approximated by separable $p_x(x)p_y(y)$ (often true in particle physics, but not always)

2D VEGAS grid

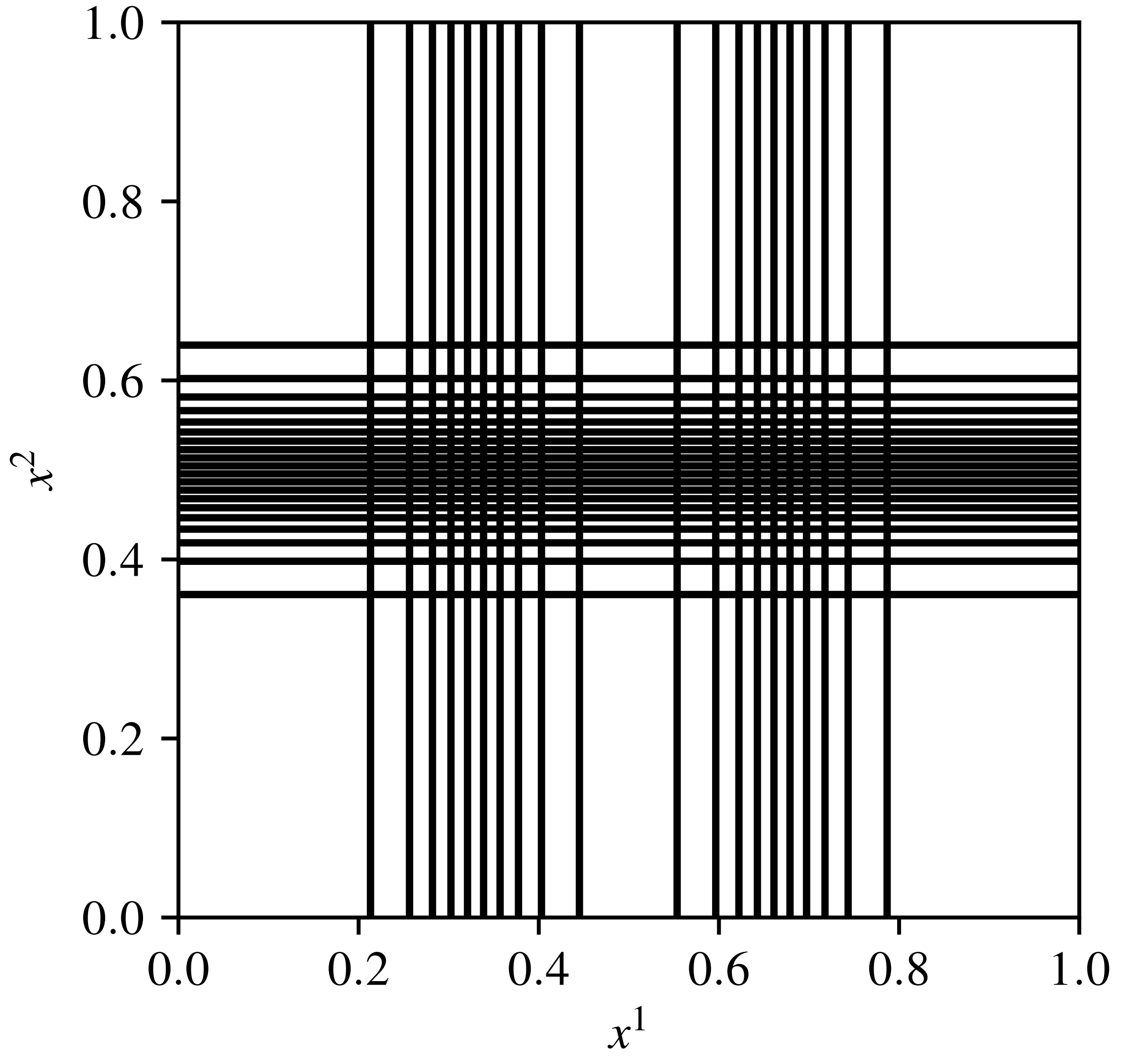


Figure 1: VEGAS map for the $D = 4$ integral defined in Eqs. (26) and Eq. (27). The figure shows grid lines for every 50th increment along the x^1 and x^2 axes; grids for the x^3 and x^4 axes are the same as for x^2.

Fig. 1 shows the grid corresponding to a VEGAS map optimized for the $D = 4$ dimensional integral

$$\int_0^1 d^4x \left(e^{-100(\mathbf{x}-\mathbf{r}_1)^2} + e^{-100(\mathbf{x}-\mathbf{r}_2)^2}\right), \tag{26}$$

where vector $\mathbf{x} = (x^1, x^2, x^3, x^4)$, and

$$\begin{aligned} \mathbf{r}_1 &= (0.33, 0.5, 0.5, 0.5) \\ \mathbf{r}_2 &= (0.67, 0.5, 0.5, 0.5). \end{aligned} \tag{27}$$

The grid concentrates increments near 0.33 and 0.67 for x^1, and near 0.5 in the other directions. Each rectangle in the figure receives, on average, the same number of Monte Carlo integration samples.

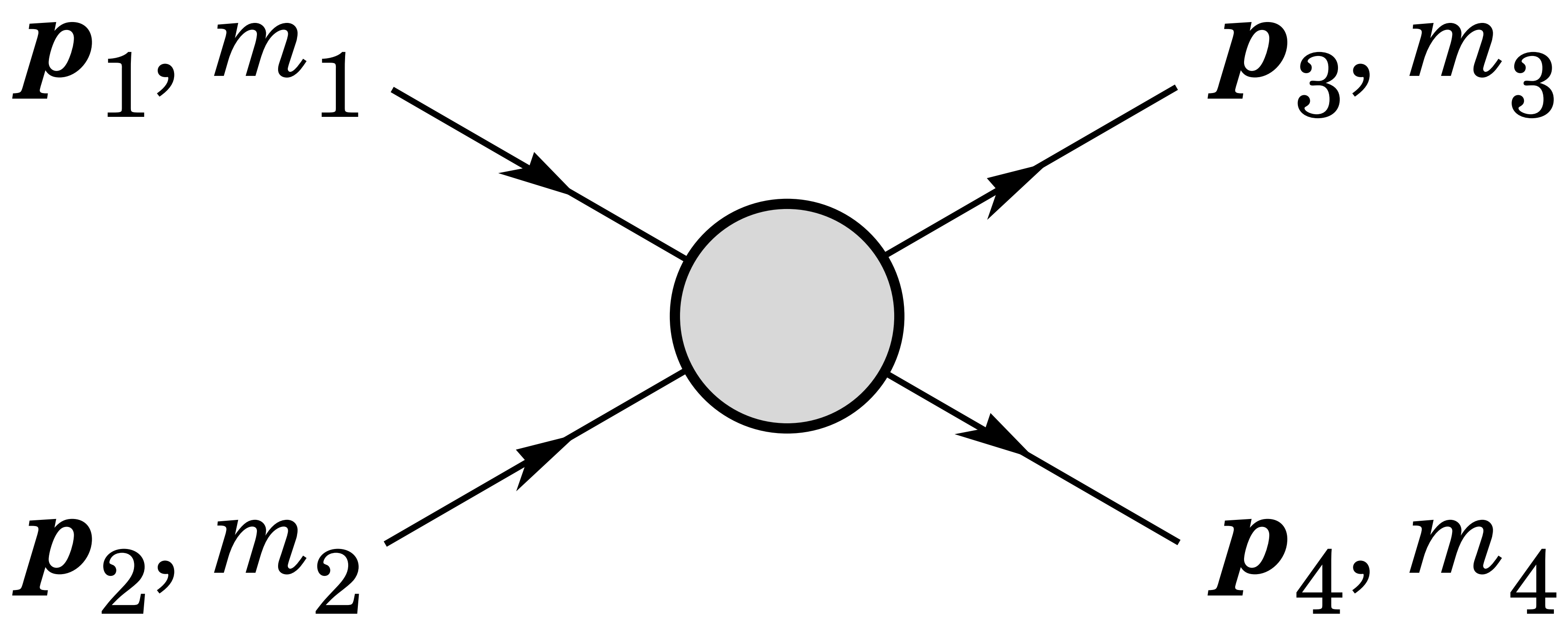
Fermi's Golden Rule (for scattering)

Ingredients: amplitude ($\mathscr{M}$) for the process and the phase space (Ω) available

For a two-to-two scattering process ($1 + 2 \to 3 + 4$), cross section is given by

$$\sigma = \frac{S}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2)^2}} \int |\mathscr{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\times \prod_{j=3}^{4} 2\pi\delta(p_j^2 - m_j^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4}$$



S accounts for double-counting with identical particles

Each outgoing particle lies on its mass shell

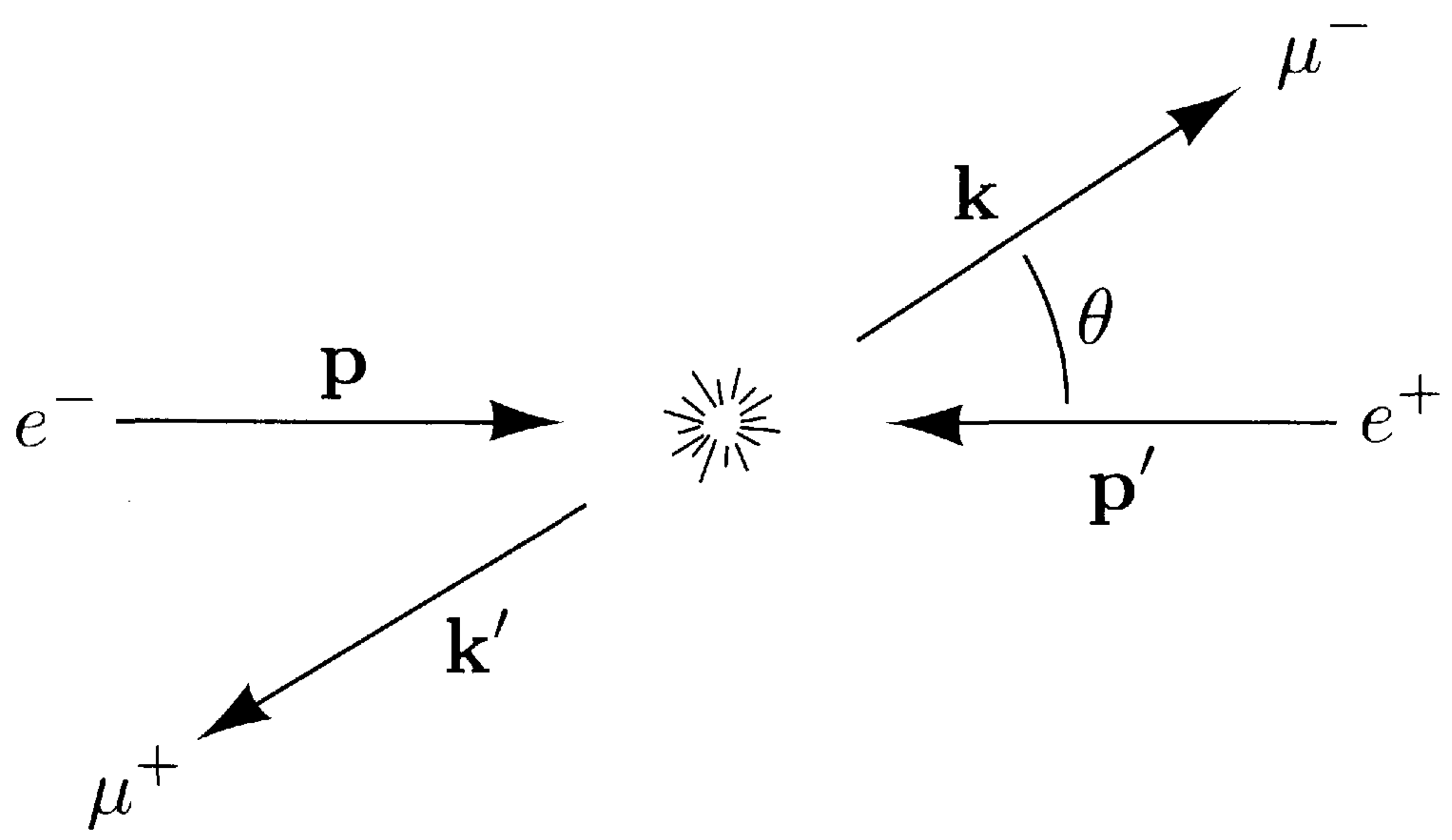
Each outgoing energy is positive

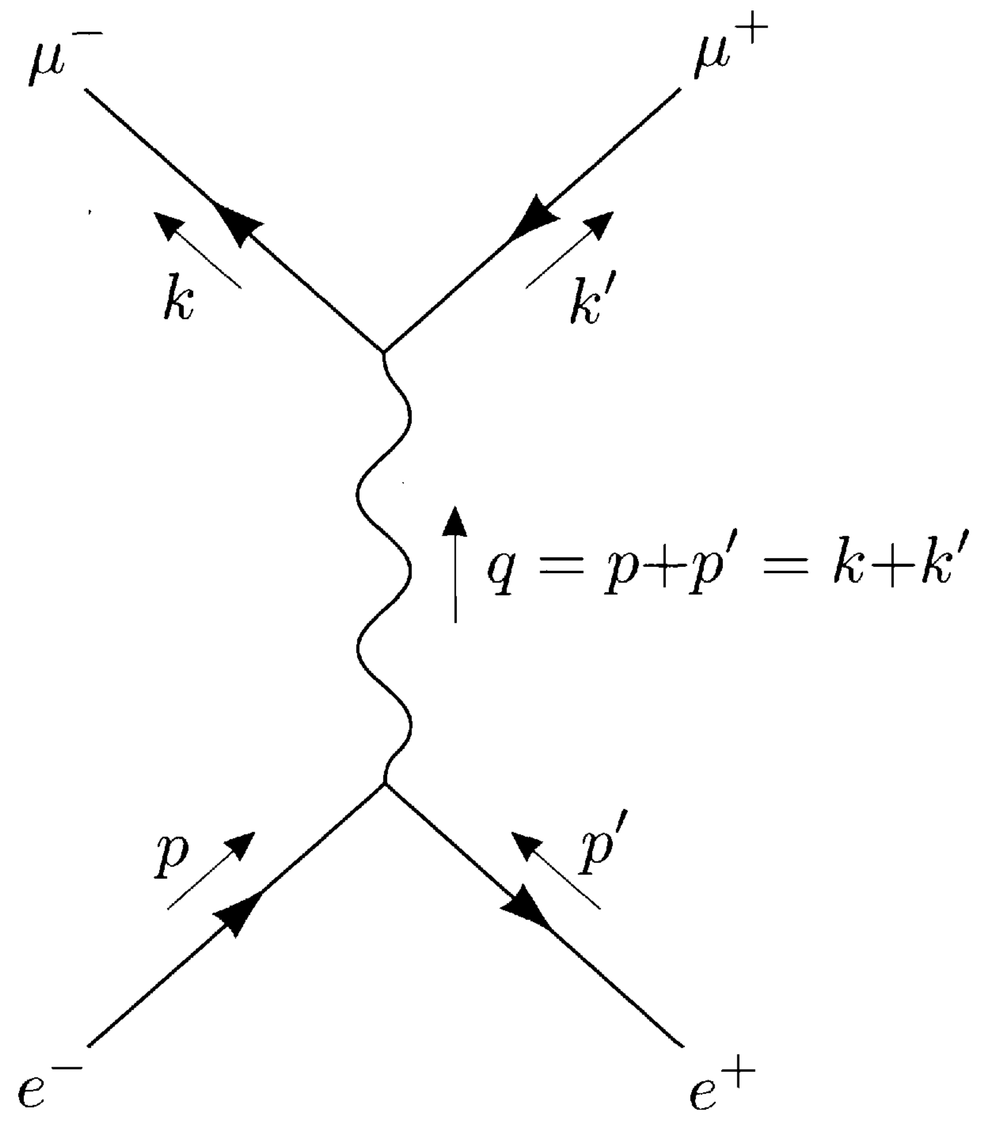
Energy and momentum must be conserved

Although we can't fully calculate σ without knowing the form of $\mathscr{M}$, we can calculate the differential cross section:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{S|\mathscr{M}|^2}{(E_1 + E_2)^2} \frac{|\boldsymbol{p}_f|}{|\boldsymbol{p}_i|}$$

$e^+e^- \to \gamma \to \mu^+\mu^-$ in Quantum Electrodynamics (QED)





- Last time, we wrote down the general differential cross section for 2-to-2 scattering
- Simplify it in the high-energy limit $E_{\rm cm} \gg m_\mu$: $$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\rm cm}^2} |\mathscr{M}|^2$$
- How do we calculate $\mathscr{M}$ in QED?

Feynman Rules for QED

Feynman Rules

1. *Notation:* To each *external* line associate a momentum $p_1, p_2, \ldots, p_n$, and draw an arrow next to the line, indicating the positive direction (forward in time).* To each *internal* line associate a momentum $q_1, q_2, \ldots$; again draw an arrow next to the line indicating the positive direction (arbitrarily assigned). See Figure 7.1.
2. *External lines:* External lines contribute factors as follows:

Electrons :	Incoming(→•) : u
	Outgoing(•→) : $\bar{u}$
Positrons :	Incoming(←•) : $\bar{v}$
	Outgoing(•←) : v
Photons :	Incoming(∿•) : ϵ_μ
	Outgoing(•∿) : $\epsilon_\mu{}^*$

3. *Vertex factors:* Each vertex contributes a factor

$$ig_e\gamma^\mu$$

The dimensionless coupling constant g_e is related to the charge of the electron: $g_e = e\sqrt{4\pi/\hbar c} = \sqrt{4\pi\alpha}$.*

4. *Propagators:* Each internal line contributes a factor as follows:

Electrons and positrons: $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$

Photons: $\dfrac{-ig_{\mu\nu}}{q^2}$

5. *Conservation of energy and momentum:* For each vertex, write a delta function of the form

$$(2\pi)^4\delta^4(k_1 + k_2 + k_3)$$

where the k's are the three four-momenta coming *into* the vertex (if an arrow leads *outward*, then k is *minus* the four-momentum of that line).
6. *Integrate over internal momenta:* For each internal momentum q, write a factor

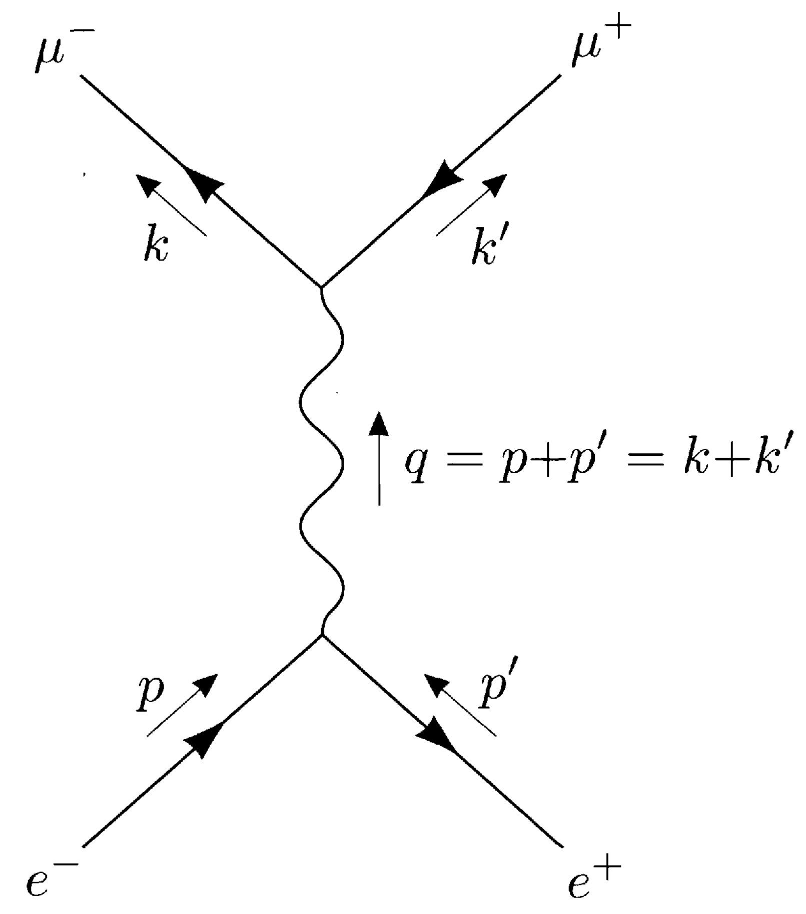
$$\frac{d^4q}{(2\pi)^4}$$

and integrate.
7. *Cancel the delta function:* The result will include a factor

$$(2\pi)^4\delta^4(p_1 + p_2 + \cdots - p_n)$$

corresponding to overall energy–momentum conservation. Cancel this factor, and multiply by i; what remains is $\mathscr{M}$.

$e^+e^- \to \gamma \to \mu^+\mu^-$ Amplitude



- Given a specific set of spins

$$-i\mathcal{M} = \bar{v}^{s'}(p')(ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(ie\gamma^\nu)v^{r'}(k')$$

- Averaging over initial spins (and summing over final spins) using completeness relations, e.g. $\sum_s u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m_e = \not{p} + m_e$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^4}\text{tr}[(\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu]\text{tr}[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu]$$

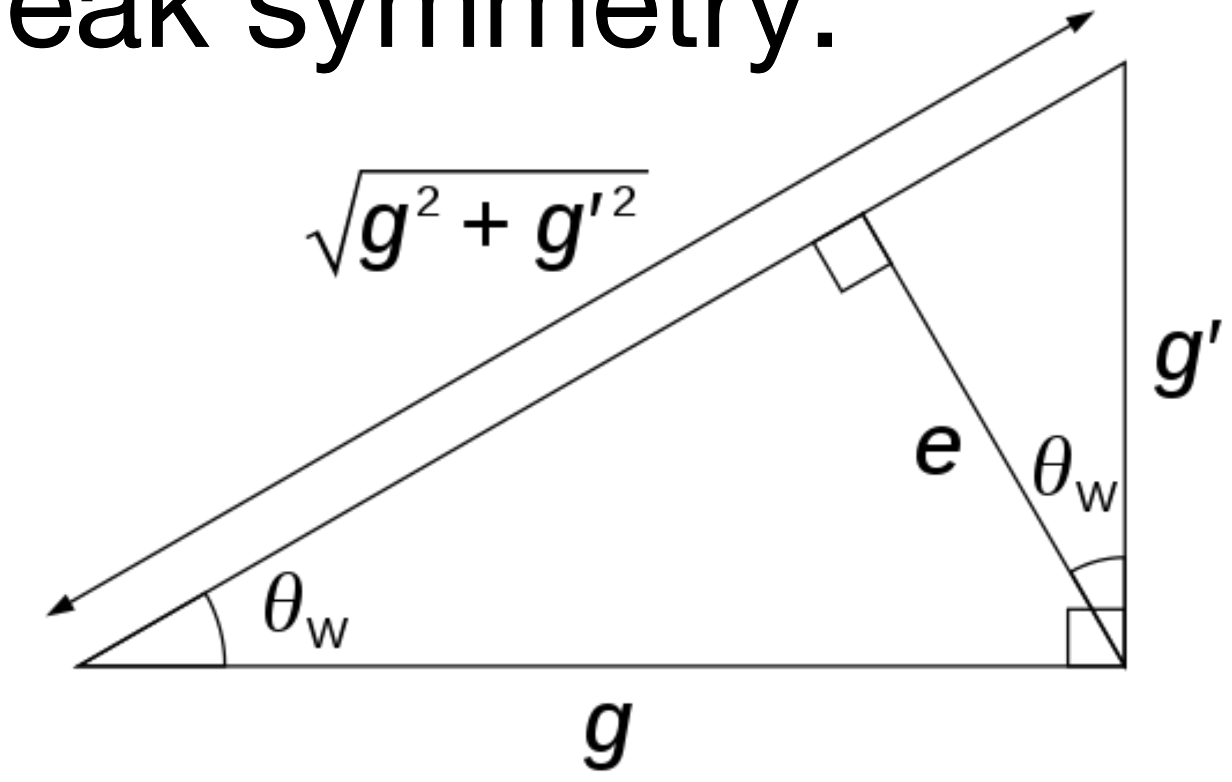
- Trace technology and high-energy limit $E_{\text{cm}} \gg m_\mu$: $\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = e^4(1 + \cos^2\theta)$

- Plugging this back into Fermi's Golden Rule gives $\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{cm}}}(1 + \cos^2\theta)$

$e^+e^- \rightarrow Z/\gamma \rightarrow \mu^+\mu^-$ Amplitude

- In the Standard Model, the W and Z bosons and the photon, are produced through the spontaneous symmetry breaking of the electroweak symmetry:

$$\begin{pmatrix} \gamma \\ Z^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B \\ W_3 \end{pmatrix},$$



- The Z boson couples differently to left-handed and right-handed fermions:

$$\mathscr{L}_{ffZ} = -\frac{g}{2\cos\theta_W}\sum_f \bar{\psi}_f \gamma^\mu (V_f - A_f\gamma_5)\psi_f Z_\mu$$

variable	symbol	value
conversion factor	$\text{GeV}^{-2} \rightarrow \text{pb}$	3.894×10^8 pb per GeV^{-2}
Z boson mass	M_Z	91.188 GeV
Z boson width	Γ_Z	2.4414 GeV
QED running coupling	α	$\frac{1}{132.507}$
Fermi constant	G_f	1.16639×10^{-5} GeV^{-2}.
Weinberg angle	$\sin^2\theta_W$	0.222246

fermions	Q_f	V_f	A_f
u, c, t	$+\frac{2}{3}$	$(+\frac{1}{2} - \frac{4}{3}\sin^2\theta_W)$	$+\frac{1}{2}$
d,s, b	$-\frac{1}{3}$	$(-\frac{1}{2} - \frac{2}{3}\sin^2\theta_W)$	$-\frac{1}{2}$
ν_e, ν_μ, ν_τ	0	$\frac{1}{2}$	$+\frac{1}{2}$
e, μ, τ	-1	$(-\frac{1}{2} + 2\sin^2\theta_W)$	$-\frac{1}{2}$

$e^+e^- \to Z/\gamma \to \mu^+\mu^-$ Cross Section

- The differential cross section is:

$$\frac{d\sigma}{d\Omega}(\hat{s}, \cos\theta) = \frac{\alpha^2}{4\hat{s}}\left[A_0(\hat{s})(1+\cos^2\theta) + A_1(\hat{s})\cos\theta\right]$$

- where the functions A_0 and A_1 are:

$$A_0(\hat{s}) = Q_e^2 - 2Q_eV_\mu V_e\ \chi_1(\hat{s}) + (A_\mu^2 + V_\mu^2)(A_e^2 + V_e^2)\ \chi_2(\hat{s})$$

$$A_1(\hat{s}) = -4Q_fA_\mu A_e\ \chi_1(\hat{s}) + 8A_\mu V_\mu A_e V_e\ \chi_2(\hat{s})$$

- And the functions χ_1 and χ_2 are:

$$\chi_1(\hat{s}) = \kappa\hat{s}(\hat{s} - M_Z^2)/((\hat{s} - M_Z^2)^2 + \Gamma_Z^2 M_Z^2)$$

$$\chi_2(\hat{s}) = \kappa^2\hat{s}^2/((\hat{s} - M_Z^2)^2 + \Gamma_Z^2 M_Z^2)$$

$$\kappa = \sqrt{2}G_F M_Z^2/(4\pi\alpha)$$

$e^+e^- \to Z/\gamma \to \mu^+\mu^-$ Cross Section

- Effect of the Z boson